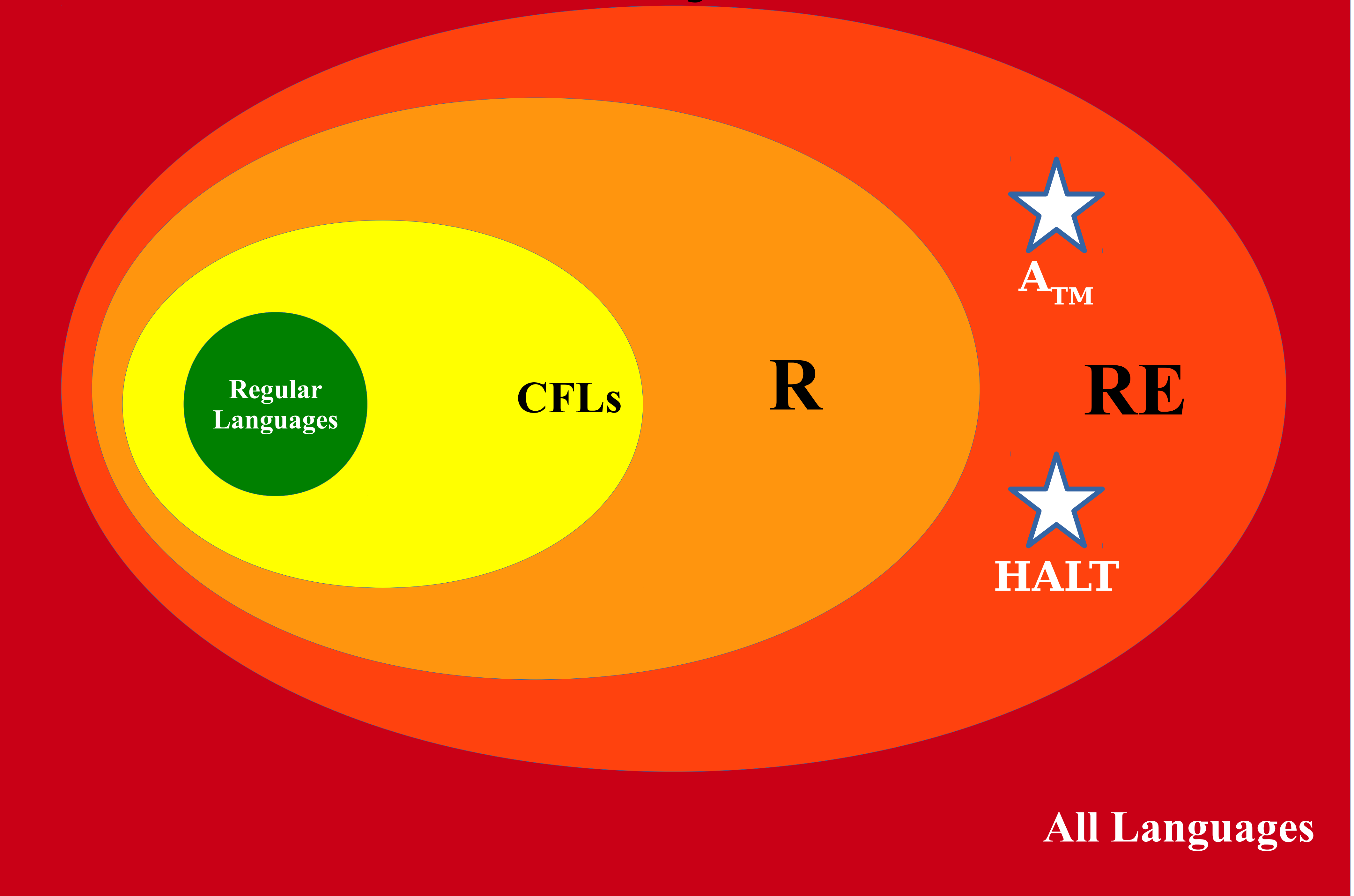
The Class RE

The Story So Far



The Class RE

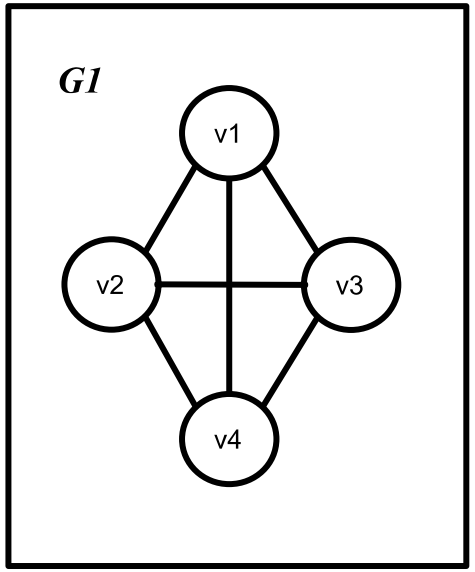
- Languages L that are in RE, *but not R*, are those where:
  - We can build a TM M, where $\mathscr{L}(M) = L$
  - That TM M has the risk of getting stuck in an infinite loop for at least some input string(s)
    - But by definition of $\mathscr{L}(M)$, only input strings that are not in L are at risk of looping in M
- Just like the class Regular was defined in multiple ways (DFAs, NFAs, RegExes), today we'll learn **another way** to define this class RE!

Get ready to answer
some questions in rapid-fire style!
(about 10 seconds per question)

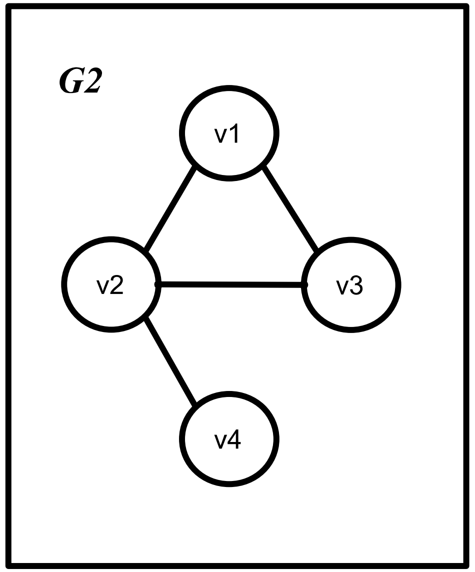
Definition:

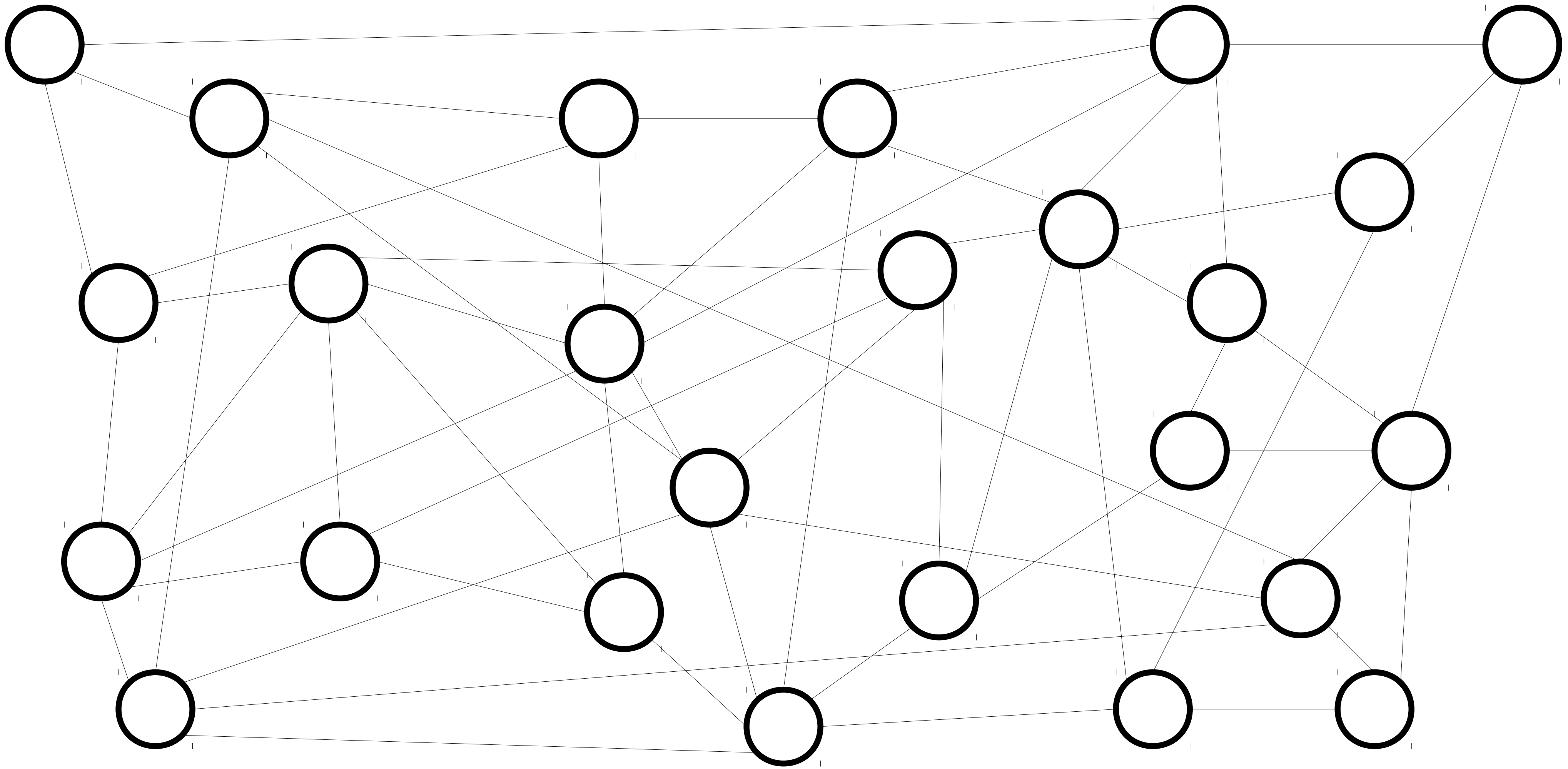
A ***k*-Clique** is a set of k vertices of a graph that are all adjacent to each other (all possible edges between those k vertices are present in the graph).

has a 4-Clique:



does *not* have a 4-Clique (has a 3-Clique though):

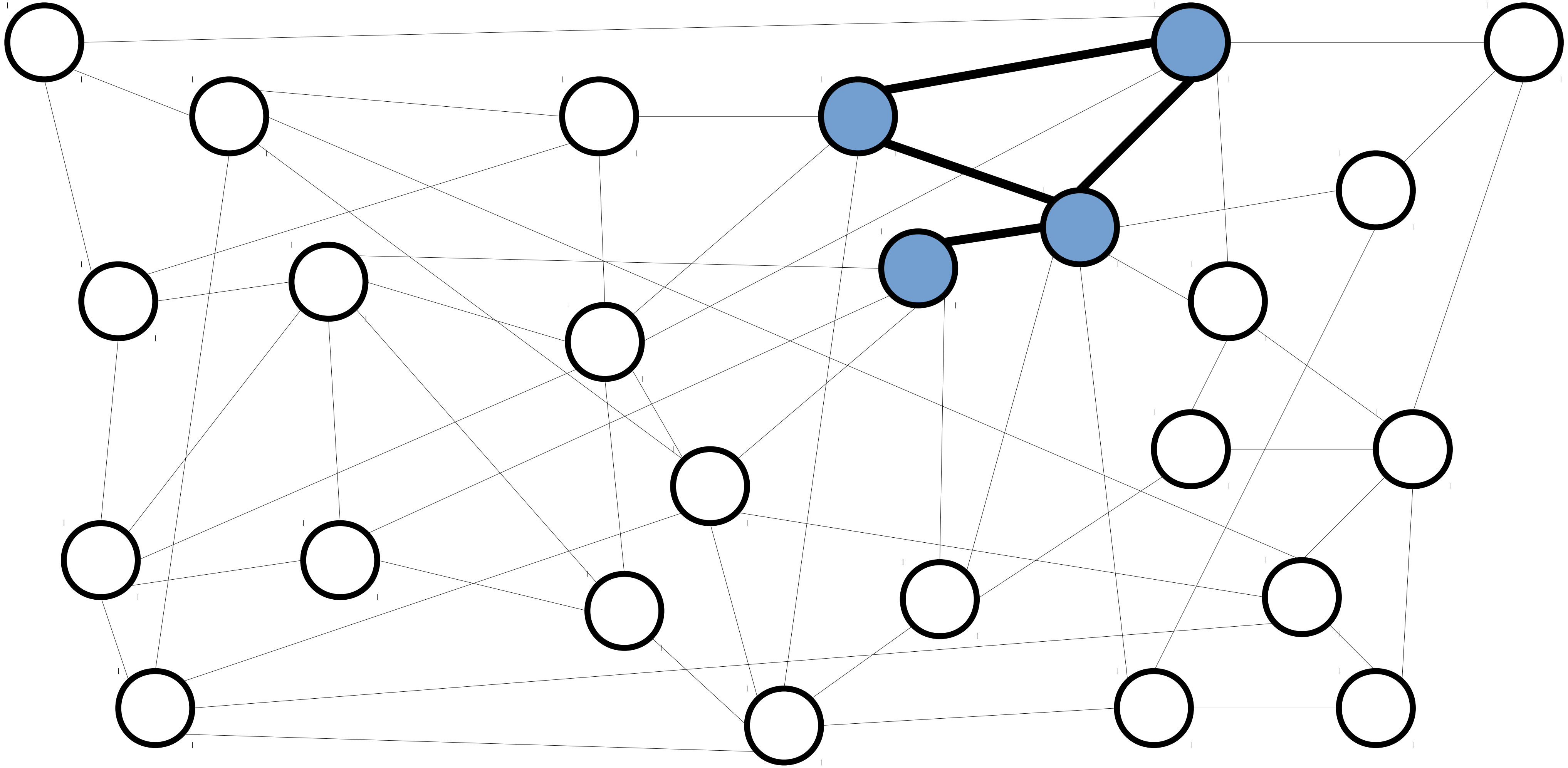




QUICK REACTION: Does this graph contain a 4-clique?

Reflection:

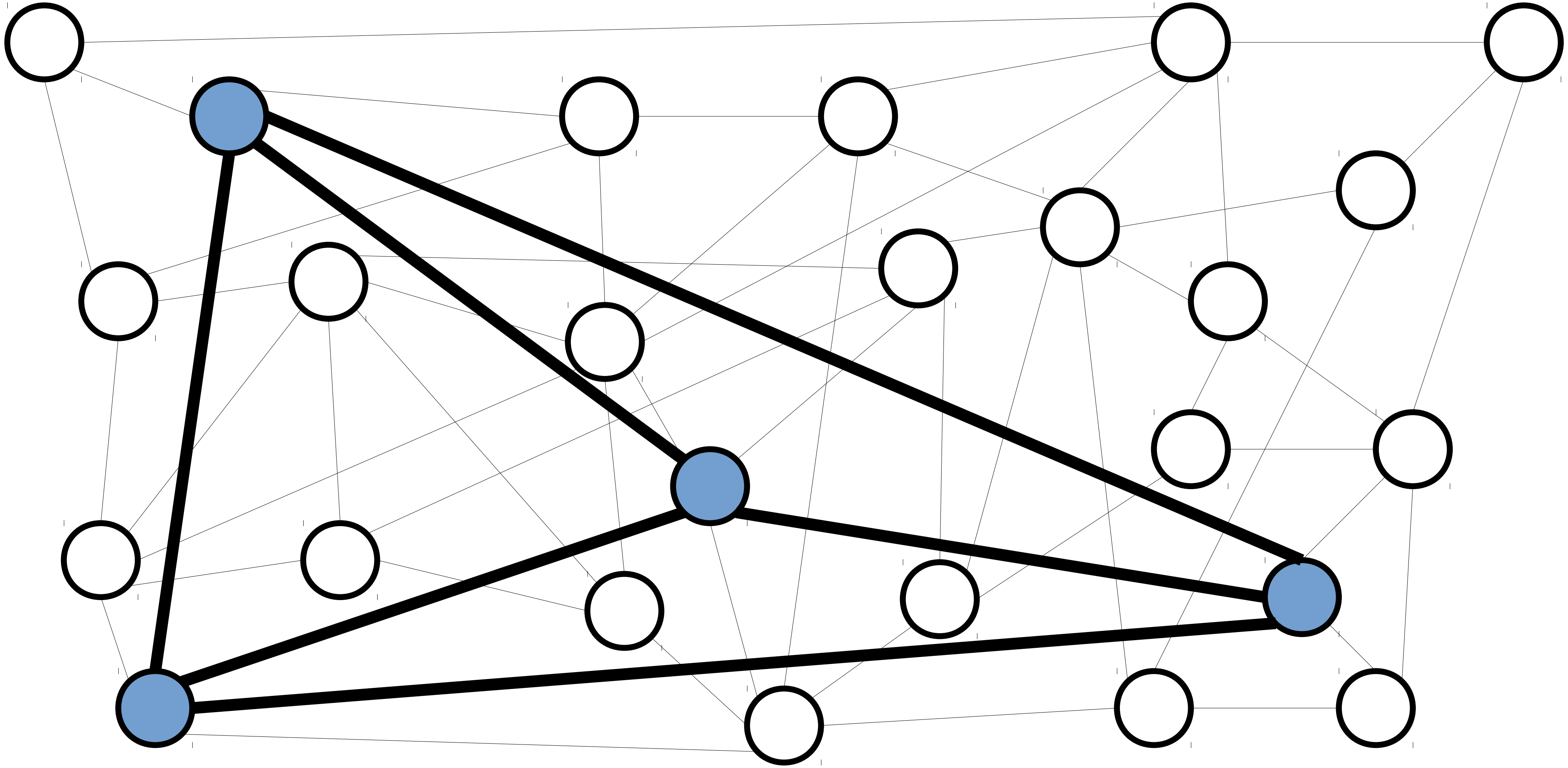
Hm, that was kind of hard to assess in just 10 seconds! What if I select and highlight just some of the nodes for you, would that be a helpful hint?



WITH A “HINT”(?): Does this graph contain a 4-clique?

Reflection:

That was a terrible so-called “hint”! It didn’t make the problem any easier to solve. :-(



WITH A *NEW* HINT: Does this graph contain a 4-clique?

Reflection:

The hint format (highlight some subset of 4 nodes) was a good format, but the hint is only helpful if the contents are the correct subset.

Discussion Question:

We found an effective, concise hint format for proving that a graph has a 4-Clique.

What about for proving a graph does **not** have a 4-Clique? What would an effective, concise hint format for that look like?

Key intuition behind our next way of defining RE:

A language L is in **RE** if, for any string w, <u>**if**</u> you know that $w \in L$, then there is some piece of evidence (a "hint") you could provide to make the problem of checking that fact very easy.

More examples of helpful hints vs unhelpful hints

Verification

		7		6		1		
					3		5	2
3			1		5	9		7
6		5		3		8		9
	1						2	
8		2		1		5		4
1		3	2		7			8
5	7		4					
		4		8		7		

Does this Sudoku puzzle have a solution?

Verification

2	5	7	9	6	4	1	8	3
4	9	1	8	7	3	6	5	2
3	8	6	1	2	5	9	4	7
6	4	5	7	3	2	8	1	9
7	1	9	5	4	8	3	2	6
8	3	2	6	1	9	5	7	4
1	6	3	2	5	7	4	9	8
5	7	8	4	9	6	2	3	1
9	2	4	3	8	1	7	6	5

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Verification

2	5	7	9	6	4	1	8	3
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3	8	6	1	2	5	9	4	7
6	4	5	7	3	2	8	1	9
7	1	9	5	4	8	3	2	6
8	3	2	6	1	9	5	7	4
1	6	3	2	5	7	4	9	8
5	7	8	4	9	6	2	3	1
9	2	4	3	8	1	7	6	5

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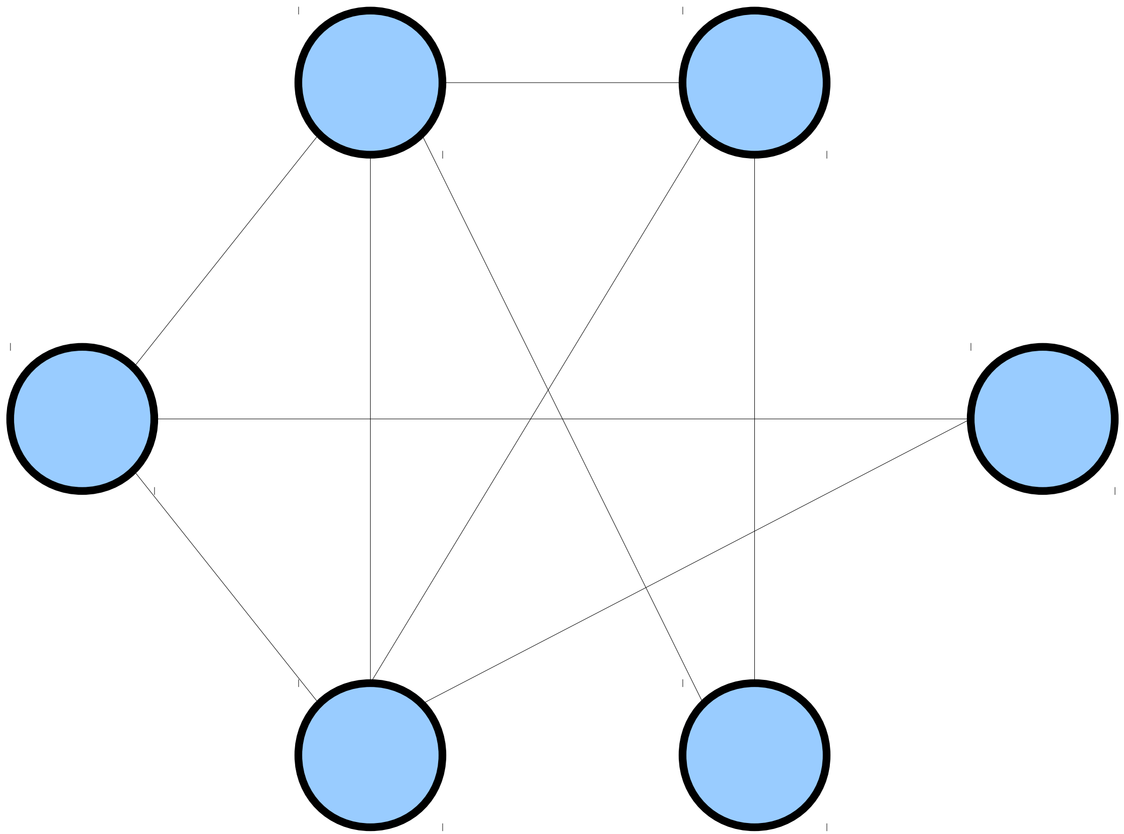
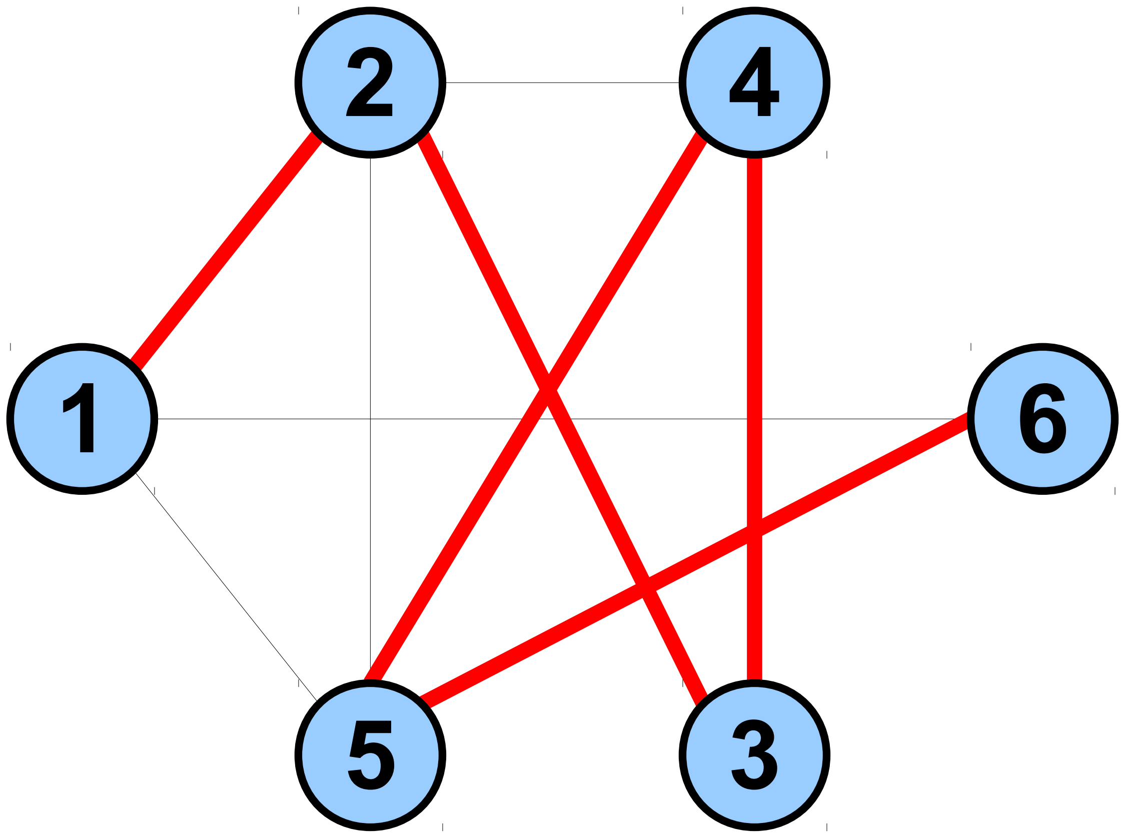
Verification



Does this graph have a ***Hamiltonian path*** (a simple path that passes through every node exactly once?)

Verification



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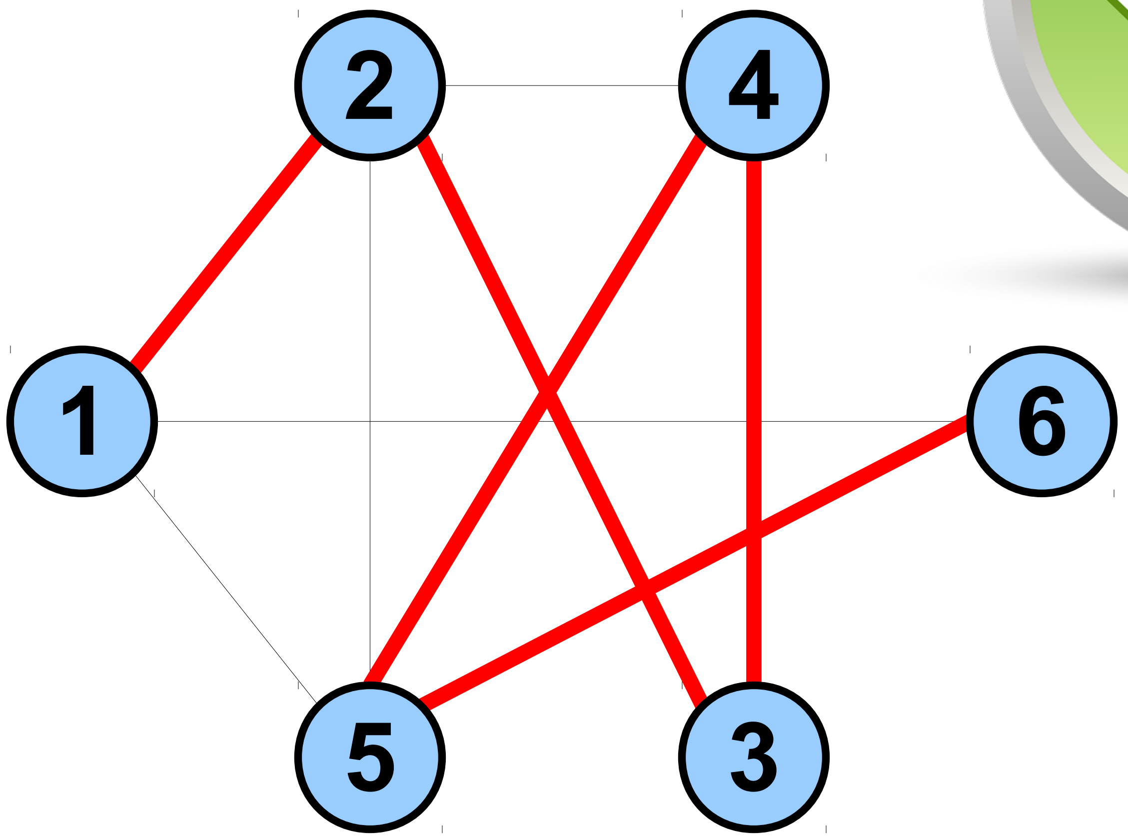

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	1						2	
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1		3	2		7			8
5	7		4					
		4		8		7		

Does this Sudoku puzzle have a solution?

Verification

1	1	7	1	6	1	1	1	1
1	1	1	1	1	3	1	5	2
3	1	1	1	1	5	9	1	7
6	1	5	1	3	1	8	1	9
1	1	1	1	4	1	1	2	1
8	1	2	1	1	1	5	1	4
1	1	3	2	1	7	1	1	8
5	7	1	4	1	1	1	1	1
1	1	4	1	8	1	7	1	1

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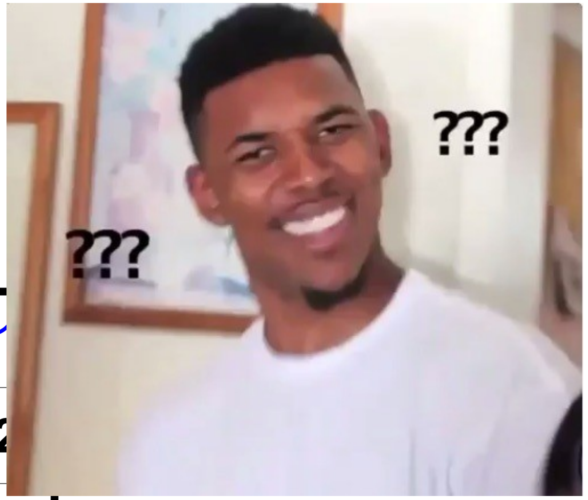
Verification



1	1	7	1	6	1	1	1	1
1	1	1	1	1	3	1	5	2
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6	1	5	1	3	1	8	1	9
1	1	1	1	4	1	1	2	1
8	1	2	1	1	1	5	1	4
1	1	3	2	1	7	1	1	8
5	7	1	4	1	1	1	1	1
1	1	4	1	8	1	7	1	1

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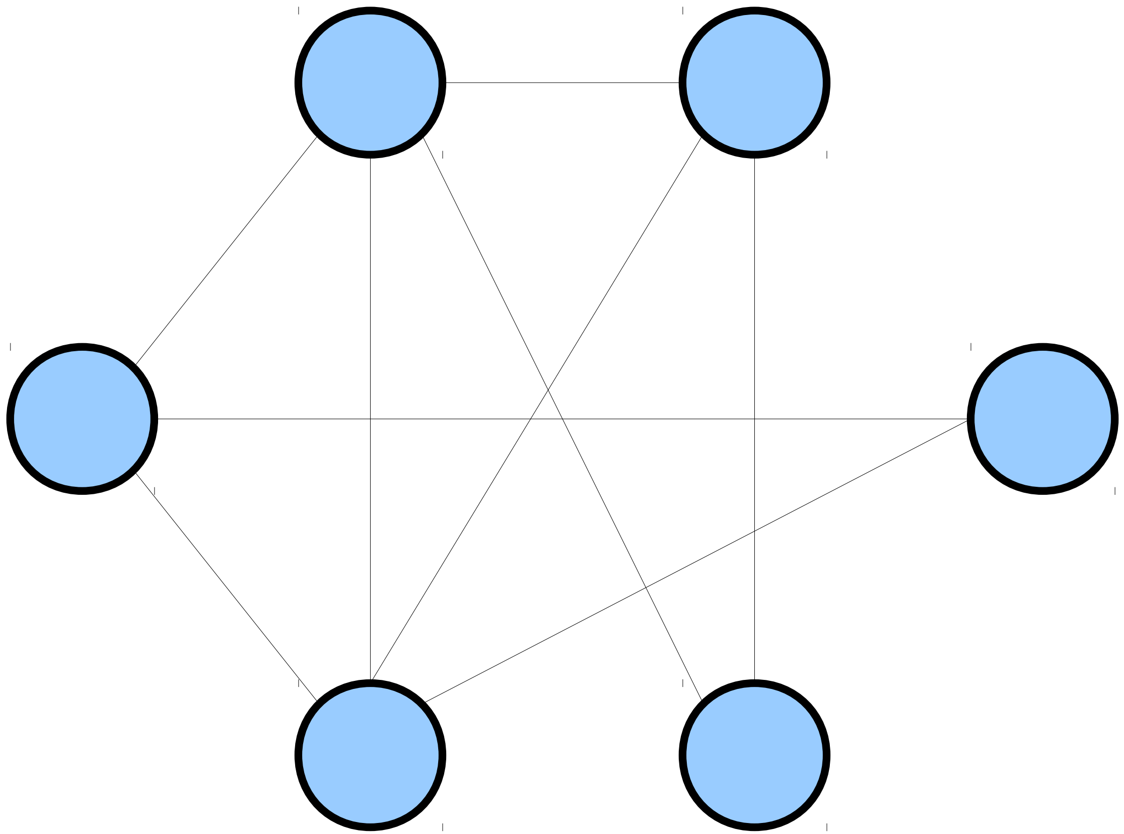
Verification



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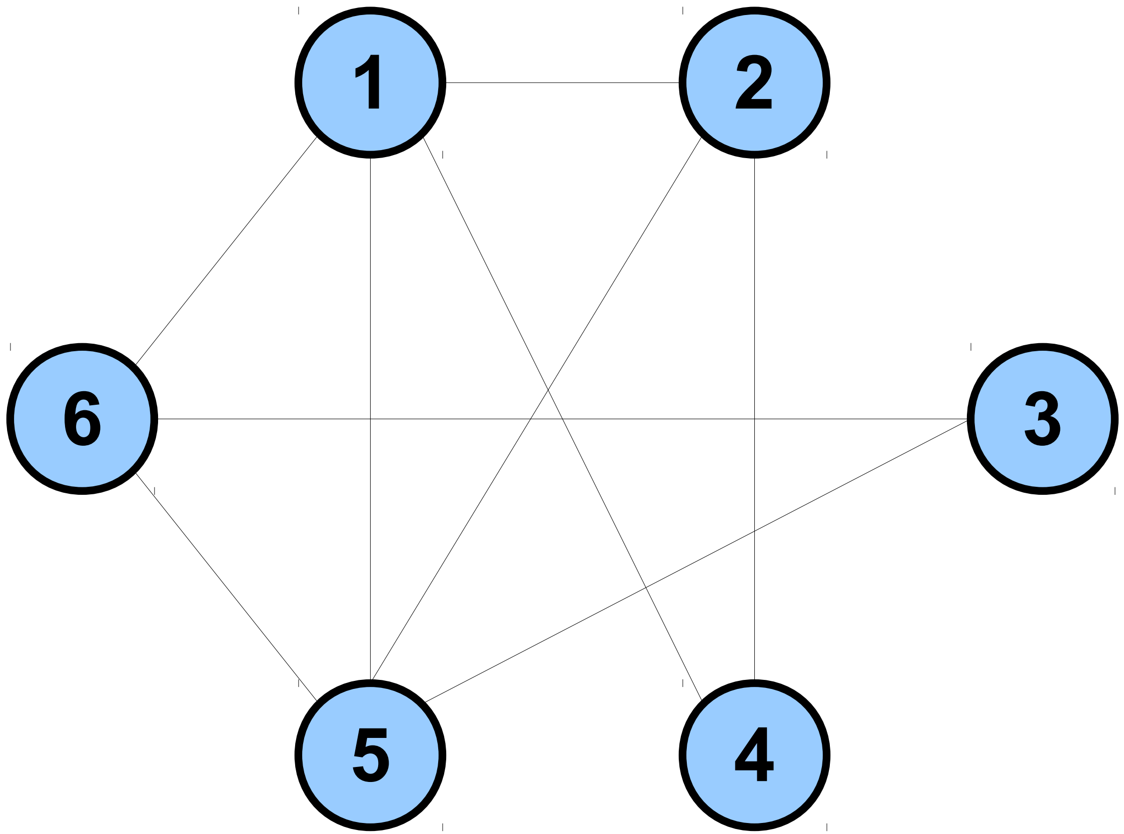

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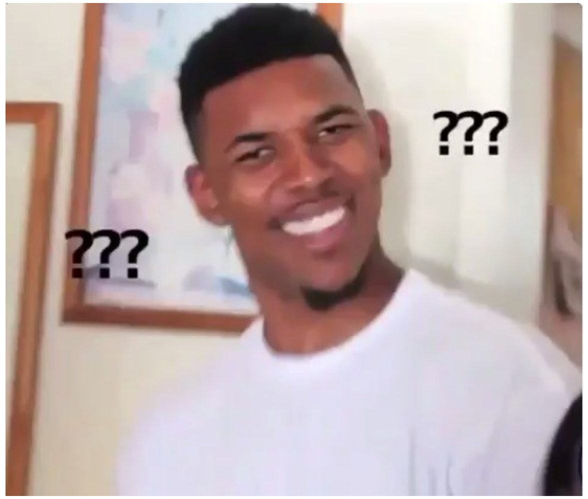
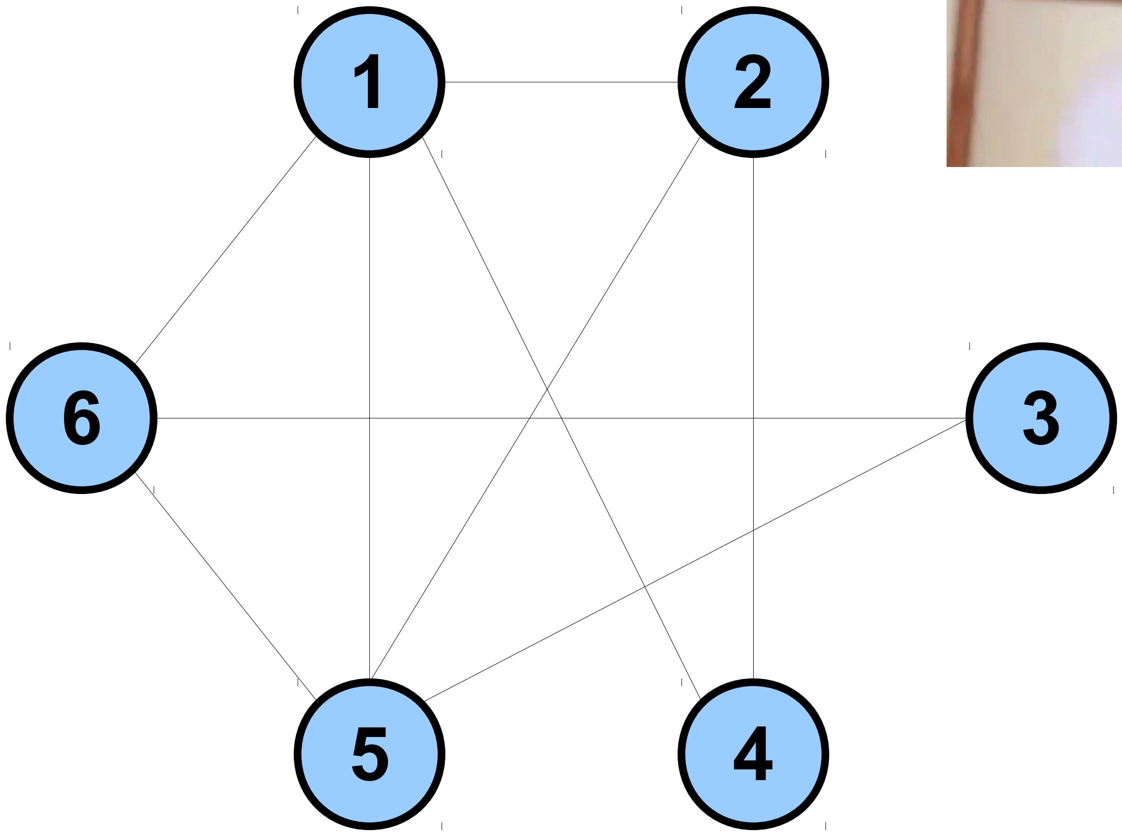




Does this graph have a ***Hamiltonian path*** (a simple path that passes through every node exactly once?)

Verification

- In each of the preceding cases, we were given some problem and some evidence supporting the claim that the answer is “yes.”
- Given correct/helpful evidence, we can quickly see that the answer is indeed “yes.”
- Given incorrect/unhelpful evidence, we aren't immediately sure whether the answer is “yes.”
  - Maybe there's *no* evidence saying that the answer is “yes,” because the answer is no!
  - Or maybe there is some evidence, but just not the evidence we were given.
- Let's formalize this idea.

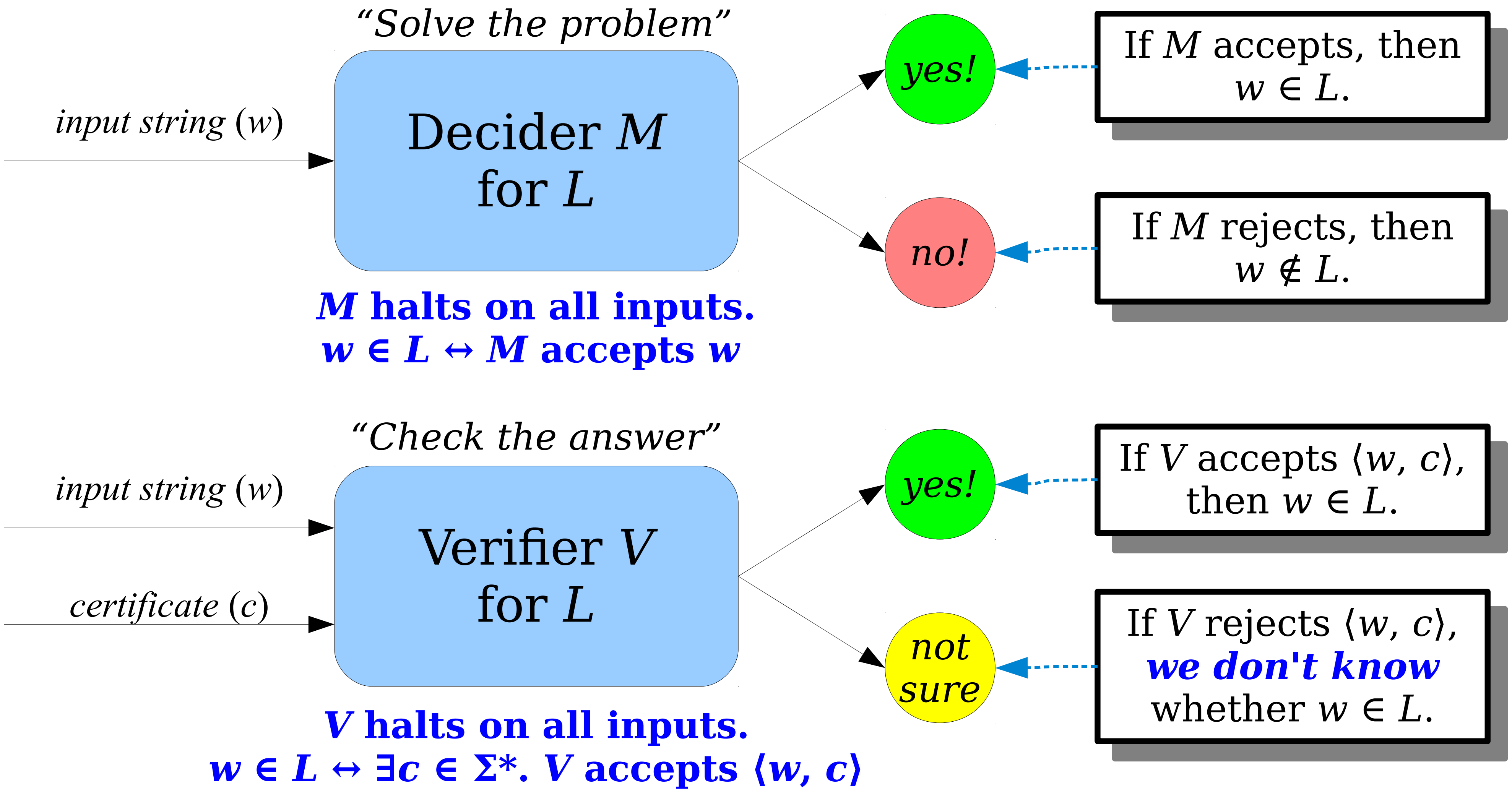
Verifiers

- A ***verifier*** for a language L is a TM V with the following properties:
  - V halts on all inputs.
  - For any string $w \in \Sigma^*$, the following is true:

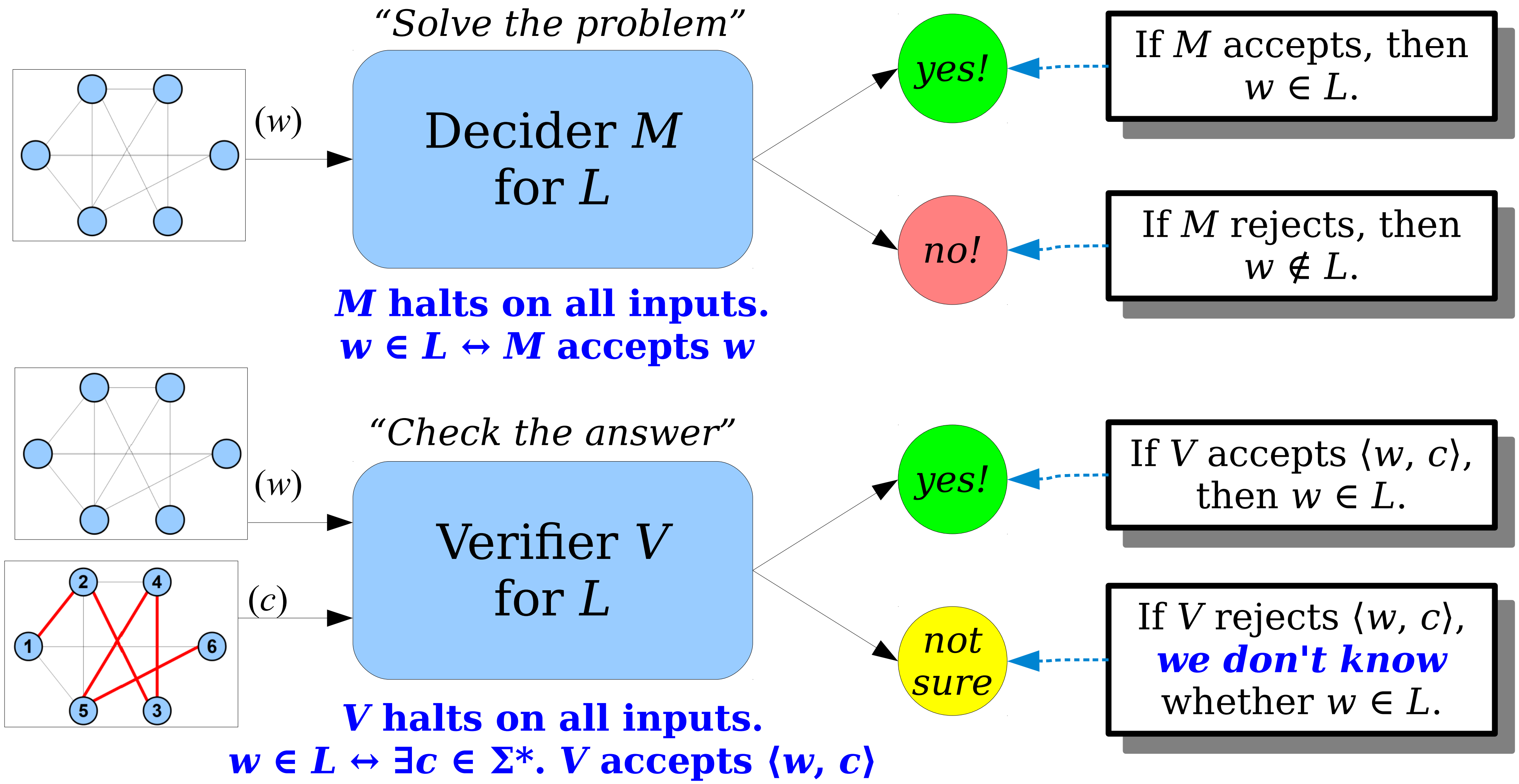
$$\boldsymbol{w \in L \quad \leftrightarrow \quad \exists c \in \Sigma^*.\ V \text{ accepts } \langle w, c \rangle}$$

- A string c where V accepts $\langle w, c \rangle$ is called a ***certificate*** for w.
  - This is the "evidence."
- Intuitively, what does this mean?

Deciders and Verifiers



Deciders and Verifiers



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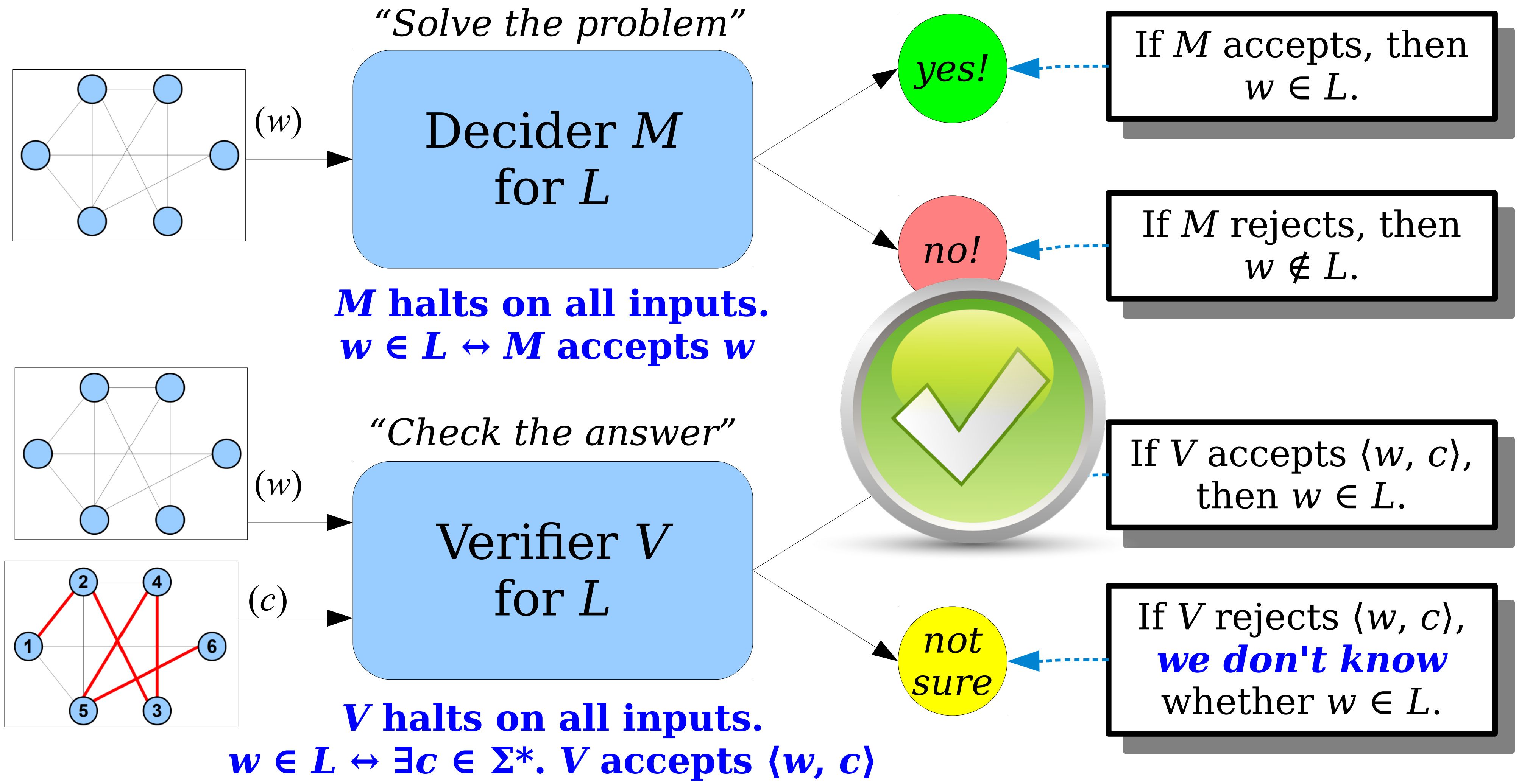
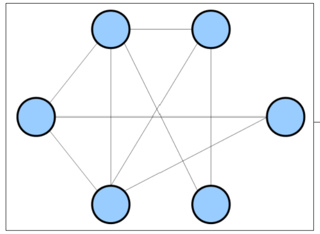


Deciders and Verifiers

"Solve the problem"



(w)

Decider M
for L

yes!

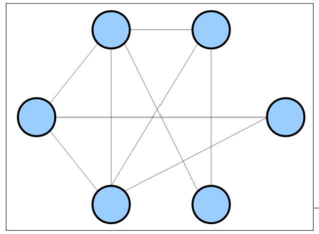
no!

If M accepts, then
$w \in L$.

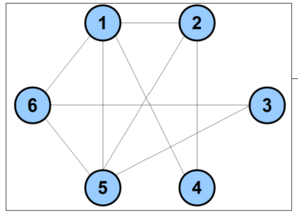
If M rejects, then
$w \notin L$.

***M* halts on all inputs.**
$w \in L \leftrightarrow M$ accepts w

"Check the answer"



(w)



(c)

Verifier V
for L

yes!

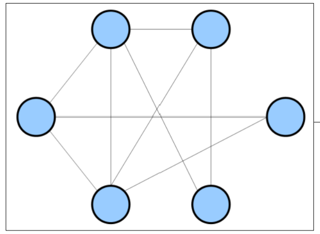
not sure

If V accepts $\langle w, c\rangle$,
then $w \in L$.

If V rejects $\langle w, c\rangle$,
we don't know
whether $w \in L$.

***V* halts on all inputs.**
$w \in L \leftrightarrow \exists c \in \Sigma^*.\ V$ accepts $\langle w, c\rangle$

Deciders and Verifiers



$\langle w \rangle$

"Solve the problem"

Decider M
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If M accepts, then
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no!

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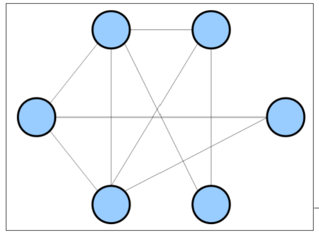
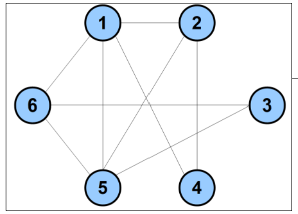


$\langle w \rangle$



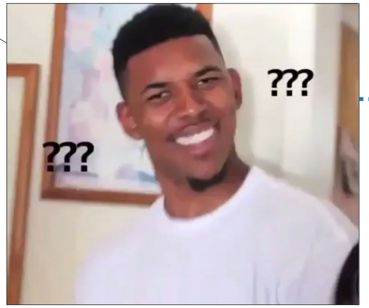
$\langle c \rangle$

"Check the answer"

Verifier V
for L

yes!

If V accepts $\langle w, c \rangle$,
then $w \in L$.



If V rejects $\langle w, c \rangle$,
we don't know
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***V* halts on all inputs.**
$w \in L \leftrightarrow \exists c \in \Sigma^*.\ V$ accepts $\langle w,$

Verifiers

- A ***verifier*** for a language L is a TM V with the following properties:
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$$\boldsymbol{w \in L \quad \leftrightarrow \quad \exists c \in \Sigma^*.\ V \text{ accepts } \langle w, c \rangle}$$

- Some notes about V:
  - If V accepts $\langle w, c \rangle$, then we're guaranteed $w \in L$.
  - If V does not accept $\langle w, c \rangle$, then either
    - $w \in L$, but you gave the wrong c, or
    - $w \notin L$, so no possible c will work.

Verifiers

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- More notes about V:
  - Notice that c is existentially quantified.
  - Notice V is required to halt *always* (like a decider).

Verifiers

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  - V halts on all inputs.
  - For any string $w \in \Sigma^*$, the following is true:

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- More notes about V:
  - Notice that $\mathscr{L}(V) \neq L$. *(Good question to hold on to for a second: what <u>is</u> $\mathscr{L}(V)$?)*
  - The job of V is just to check certificates, not to decide membership in L.

Verifiers

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- A note about c:
  - Figuring out what would make a good certificate (should it be a number of steps to take, an equation-solving variable assignment, a set of graph nodes, an array of numbers to fill in a whole Sudoku board?) is custom work to do for each different language L.

Some Verifiers

- Let L be the following language:

$$L = \{ \langle n \rangle \mid n \in \mathbb{N} \text{ and the hailstone sequence terminates for } n \}$$

```
bool checkHailstone(int n, int c) {
    for (int i = 0; i < c; i++) {
        if (n % 2 == 0) n /= 2;
        else n = 3*n + 1;
        if (n == 1) return true;
    }
    return n == 1;
}
```

Some Verifiers

Does this always halt?

$L = \{ \langle n \rangle \mid n \in \mathbb{N}$ and the hailstone sequence terminates for n $\}$

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bool checkHailstone(int n, int c) {
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```

Some Verifiers

For one given $\langle n \rangle \in L$ (say 11), how many different values of c will work to cause the verifier to accept?

$L = \{ \langle n \rangle \mid n \in \mathbb{N}$ and the hailstone sequence terminates for n $\}$

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bool checkHailstone(int n, int c) {
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    }
    return n == 1;
}
```

How many of these statements are true of $\mathscr{L}(V)$?

- $\mathscr{L}(V) = L$
- $\mathscr{L}(V) \subseteq L$
- $L \subseteq \mathscr{L}(V)$

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}
```

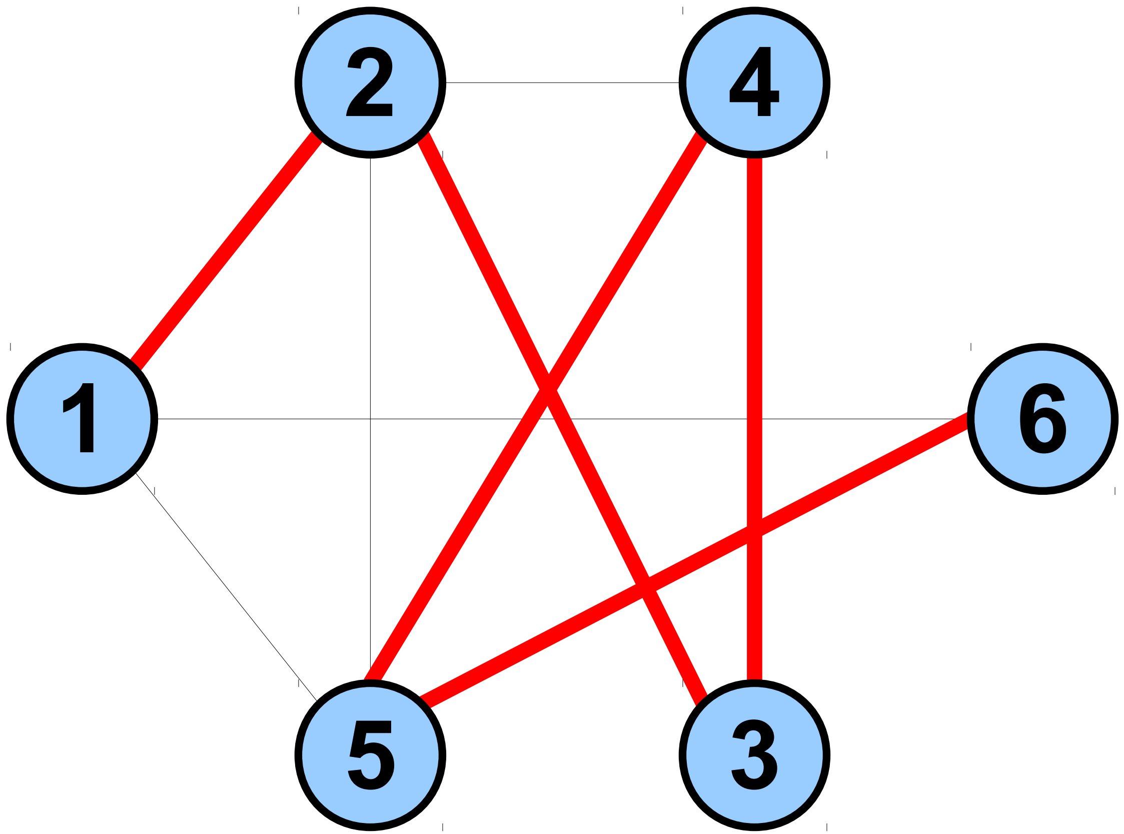
Some Verifiers

- Let L be the following language:

  $$L = \{ \langle G \rangle \mid G \text{ is a graph and } G \text{ has a Hamiltonian path } \}$$

- (A Hamiltonian path is a simple path that visits every node in the graph.)
- Let's see how to build a verifier for L.

Verification



Is there a simple path that goes
through every node exactly once?

Verifier Example: Hamiltonian Path

- Let L be the following language:

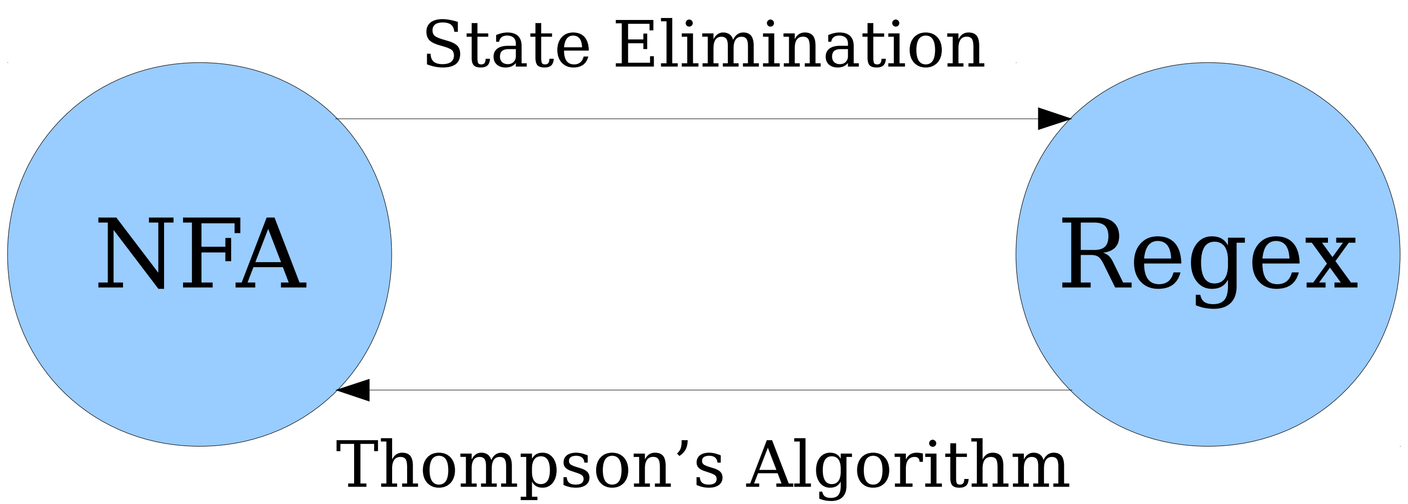
$$L = \{ \langle G \rangle \mid G \text{ is a graph with a Hamiltonian path } \}$$

```
bool checkHamiltonian(Graph G, vector<Node> c) {
   if (c.size() != G.numNodes()) return false;
   if (containsDuplicate(c)) return false;

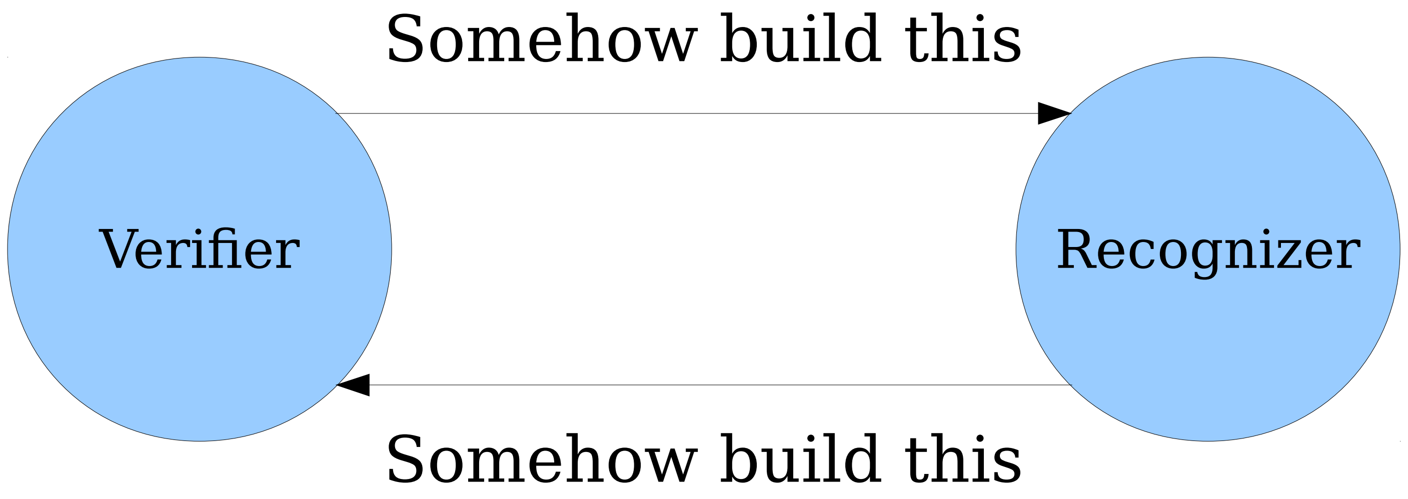
   for (size_t i = 0; i < c.size() - 1; i++) {
       if (!G.hasEdge(c[i], c[i+1])) return false;
   }
   return true;
}
```

- Do you see why $\langle G \rangle \in L$ iff there is a c where `checkHamiltonian(G, c)` returns true?
- Do you see why `checkHamiltonian` always halts?

Where We've Been



Where We're Going Today



Verifier for A_{TM}?

- Consider A_{TM}:

  $A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \}.$

- This is our standard example of an undecidable language. There's no way, in general, to tell whether a TM *M* will accept a string *w*.
- Although this language is undecidable, it's an **RE** language, and it's possible to build a verifier for it!

What would make a good certificate for a verifier for A_{TM}?

- Consider A_{TM}:

$$A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \}.$$

- This is a ***canonical*** example of an undecidable language. There's no way, in general, to tell whether a TM M will accept a string w.
- Although this language is undecidable, it's an **RE** language, and it's possible to build a verifier for it!

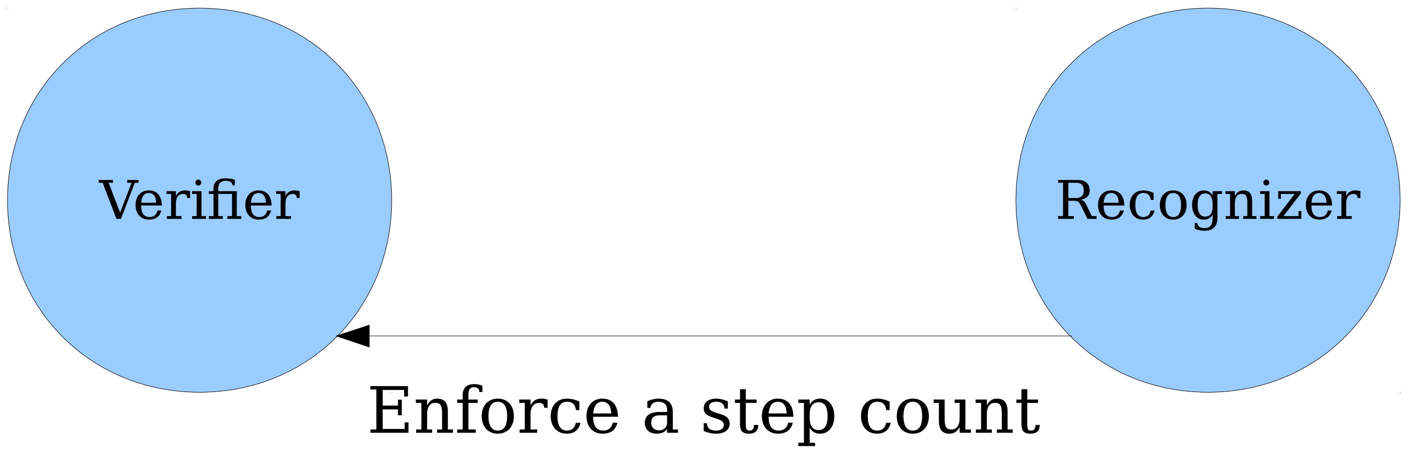
A Verifier for A_{TM}

- Recall A_{TM} = { ⟨*M*, *w*⟩ | *M* is a TM and *M* accepts *w*}

```
bool checkWillAccept(TM M, string w, int c) {
    set up a simulation of M running on w;
    for (int i = 0; i < c; i++) {
       simulate the next step of M running on w;
    }
    return whether M is in an accepting state;
}
```

- Do you see why *M* accepts *w* iff there is some *c* such that `checkWillAccept(M, w, c)` returns true?
- Do you see why `checkWillAccept` always halts?

Equivalence of Verifiers and Recognizers



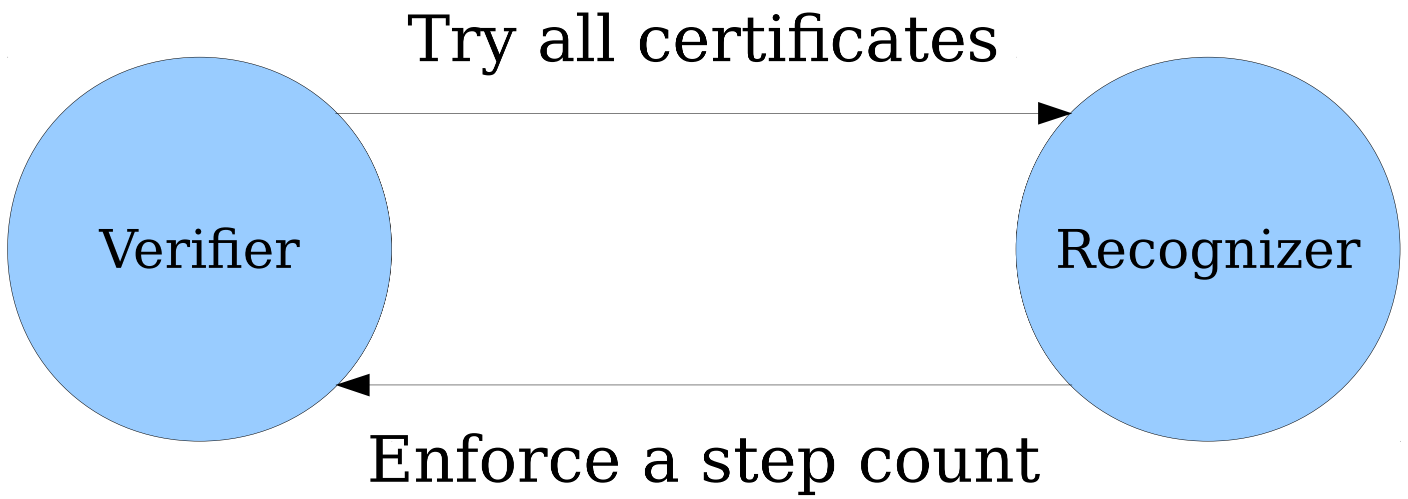
What languages are verifiable?

Let V be a verifier for a language L. Consider the following function given in pseudocode:

```
bool mysteryFunction(string w) {
    int i = 0;
    while (true) {
        for (each string c of length i) {
            if (V accepts ⟨w, c⟩) return true;
        }
        i++;
    }
}
```

What set of strings does `mysteryFunction` return **true** on?

Equivalence of Verifiers and Recognizers



Theorem: If L is a language, then there is a verifier for L if and only if $L \in \mathbf{RE}$.

Verifiers and **RE**

- ***Theorem:*** If there is a verifier V for a language L, then $L \in$ **RE**.
- ***Proof goal:*** Given a verifier V for a language L, find a way to construct a recognizer M for L.

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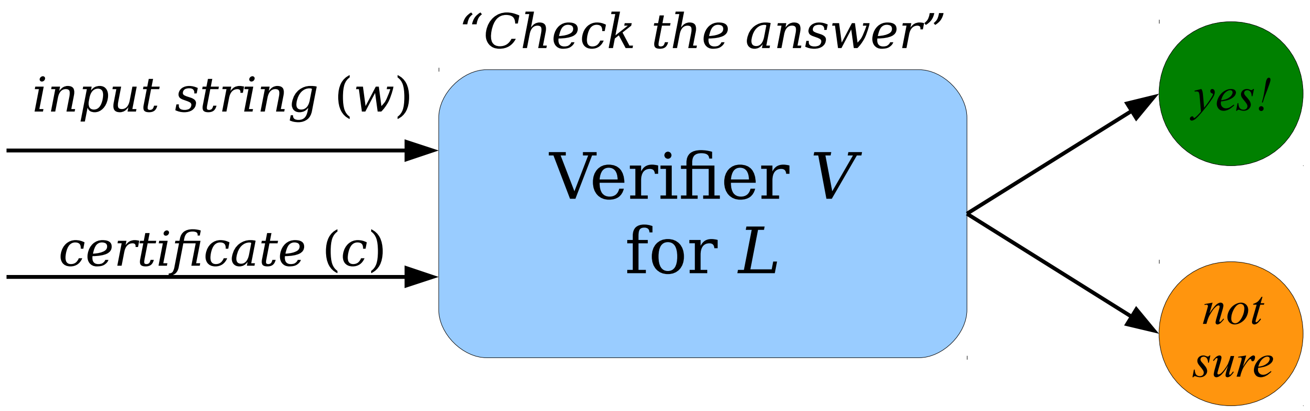

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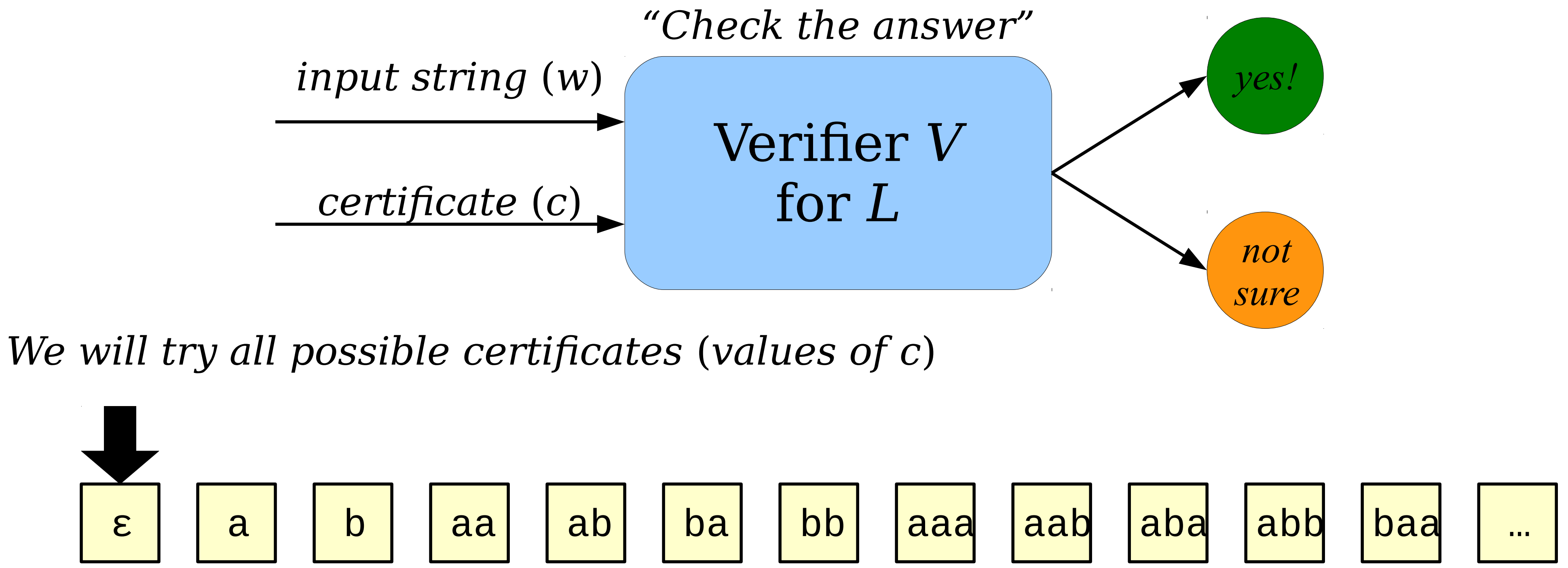

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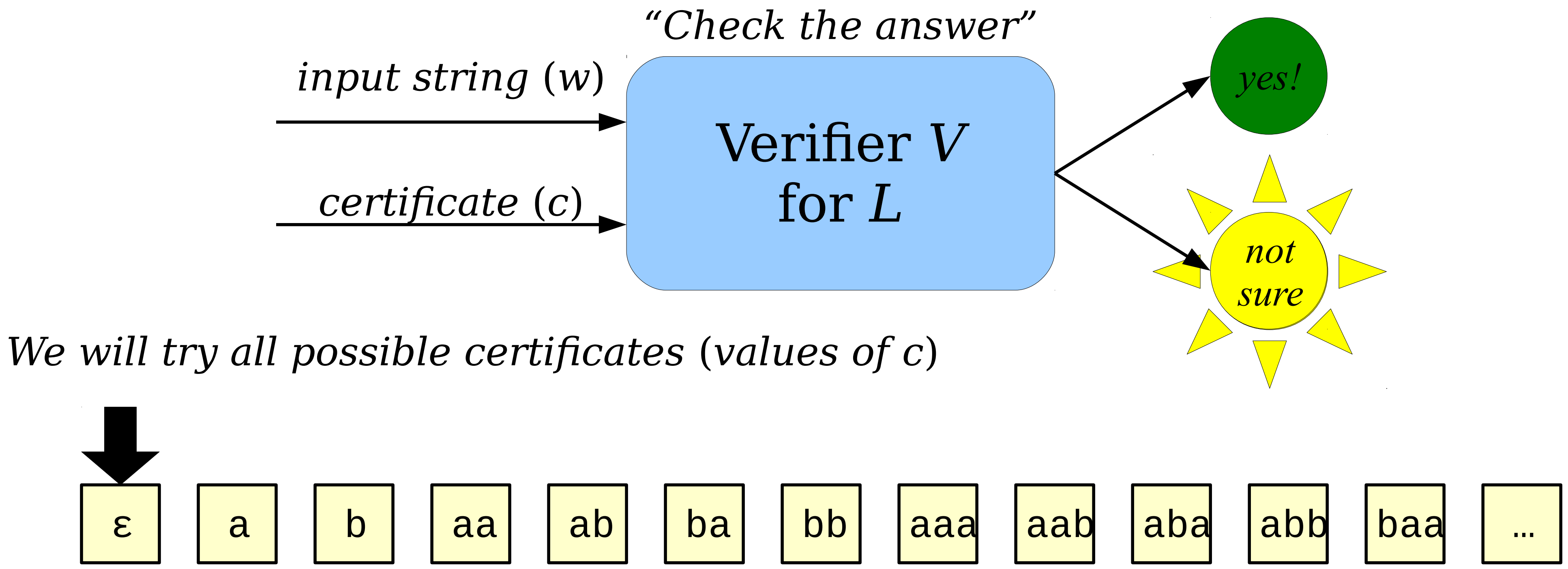

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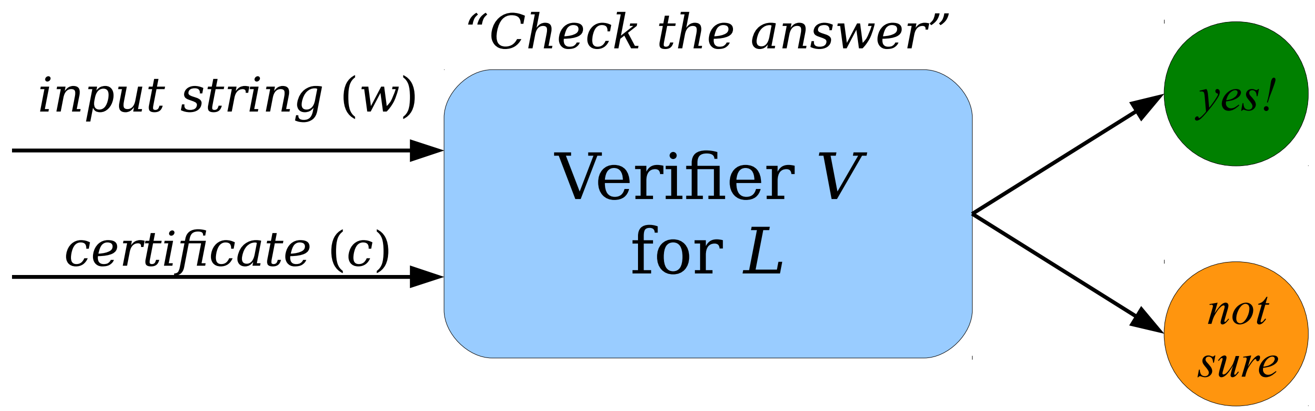

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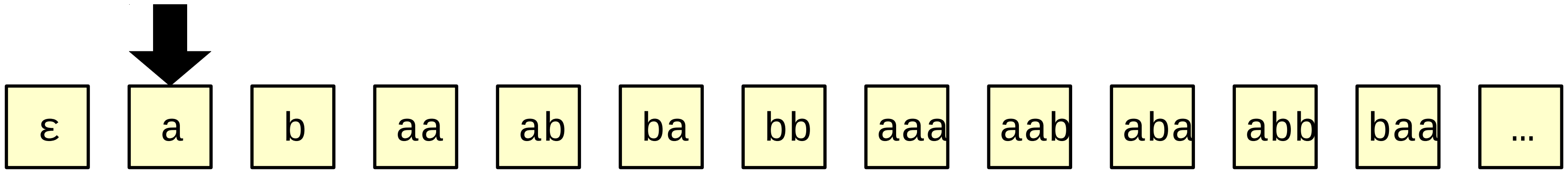

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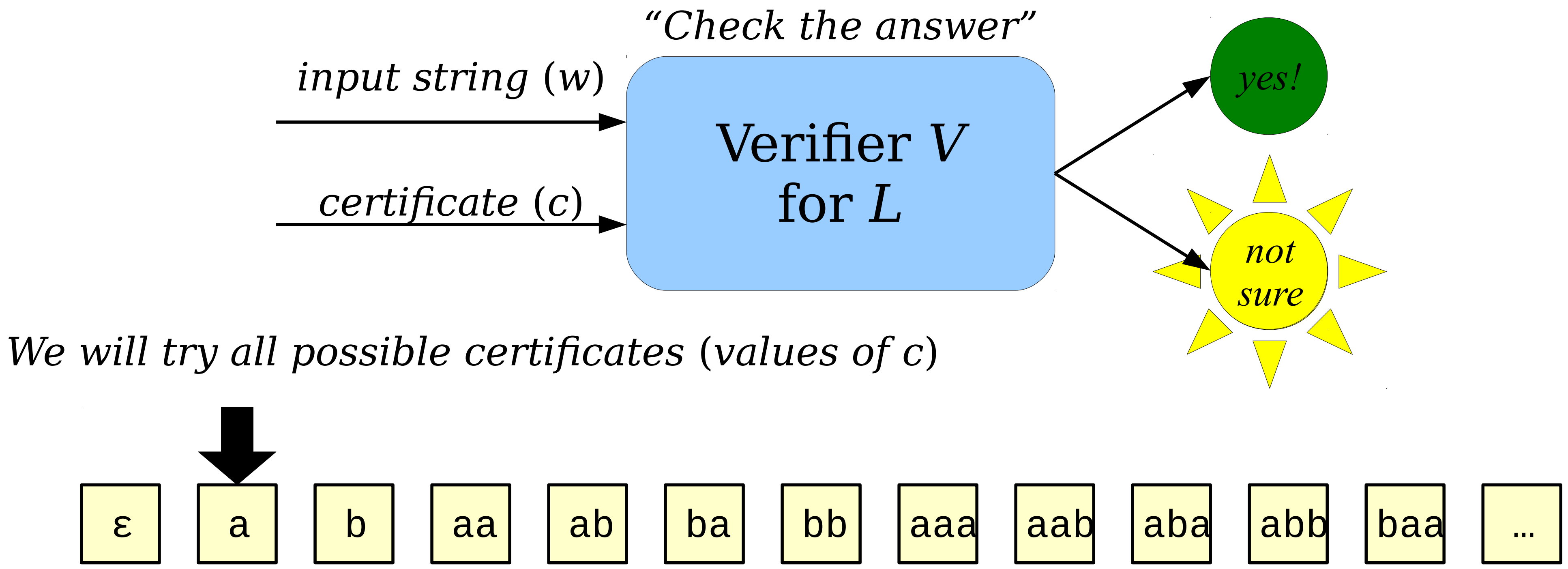

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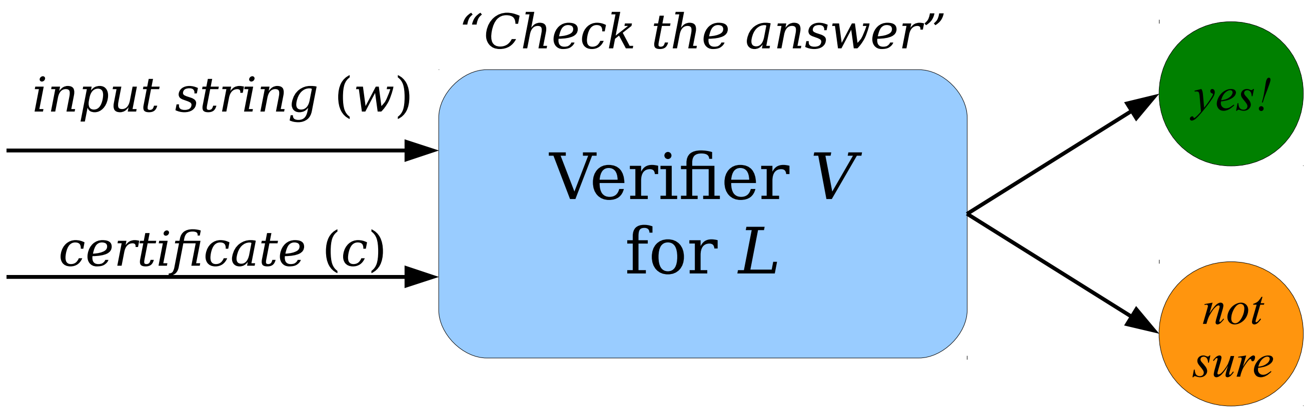

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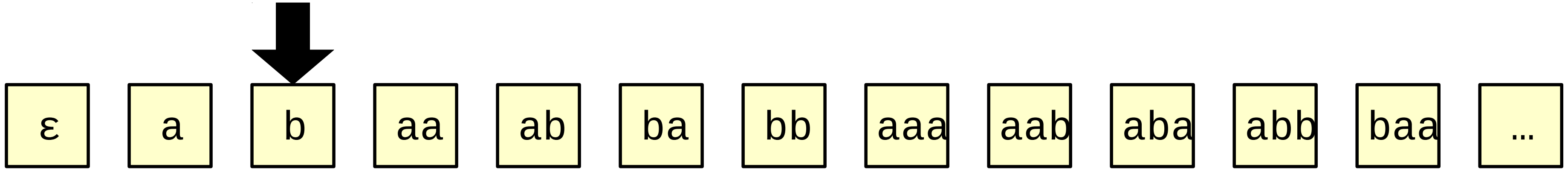

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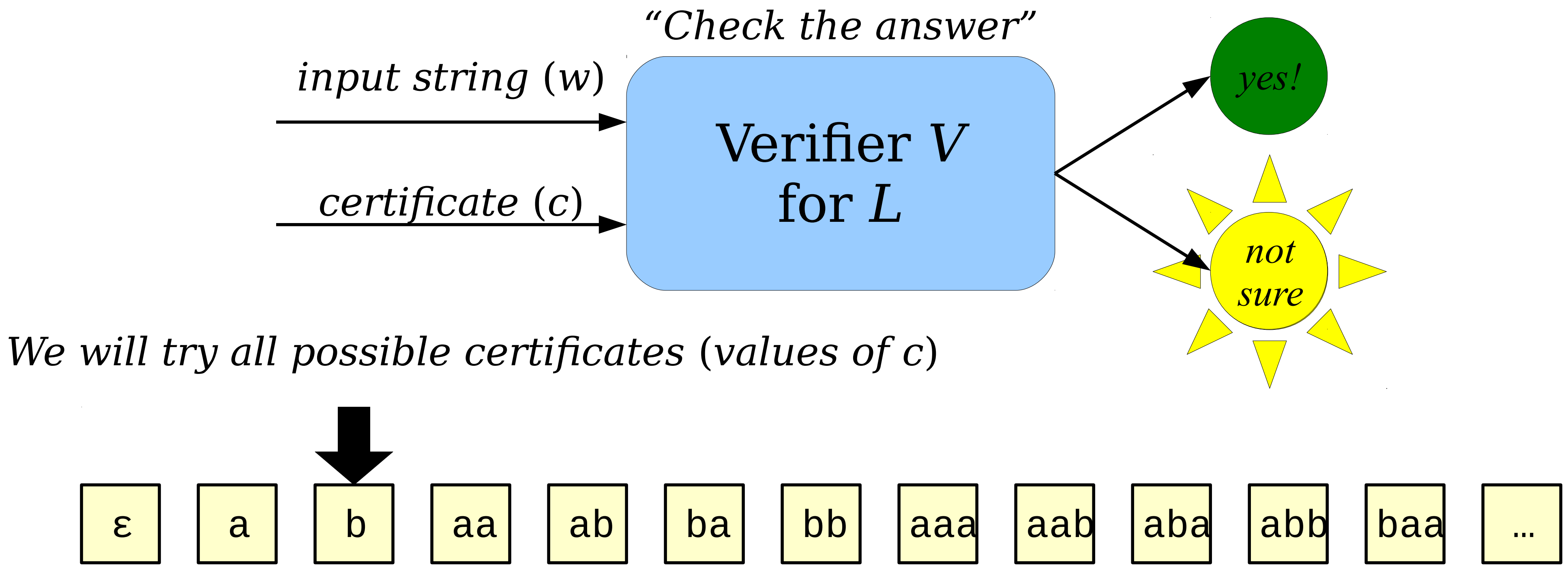

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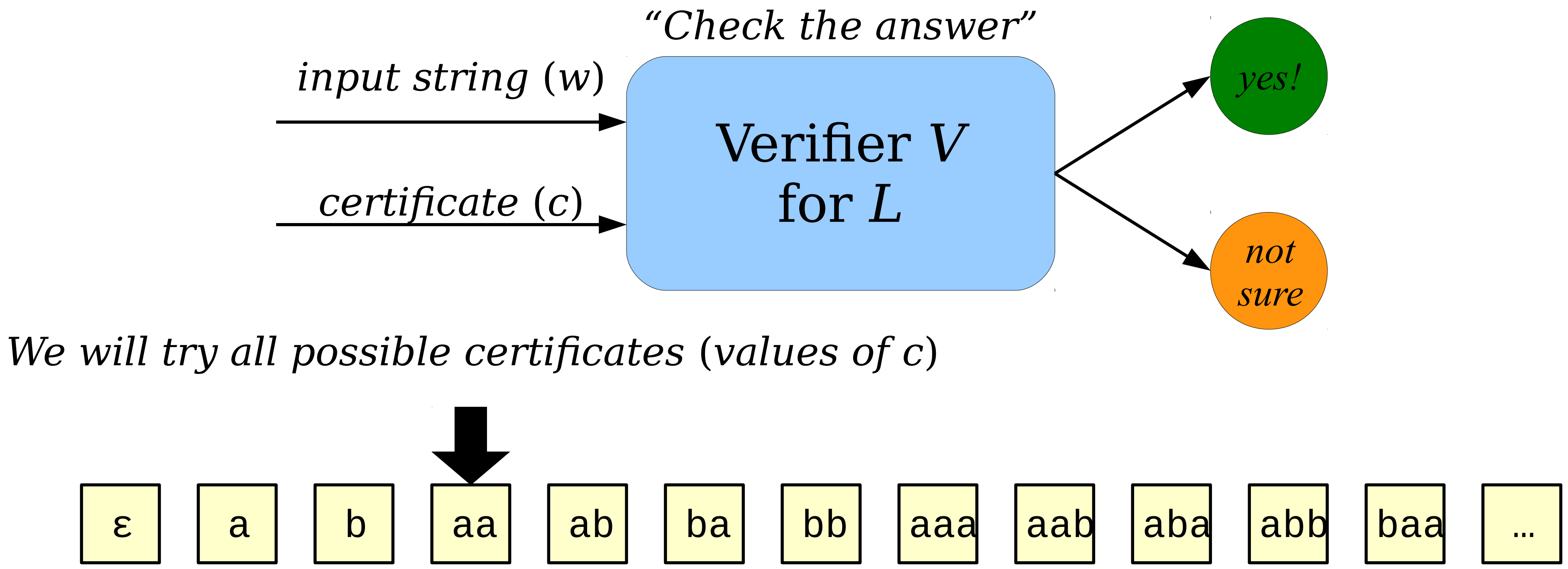

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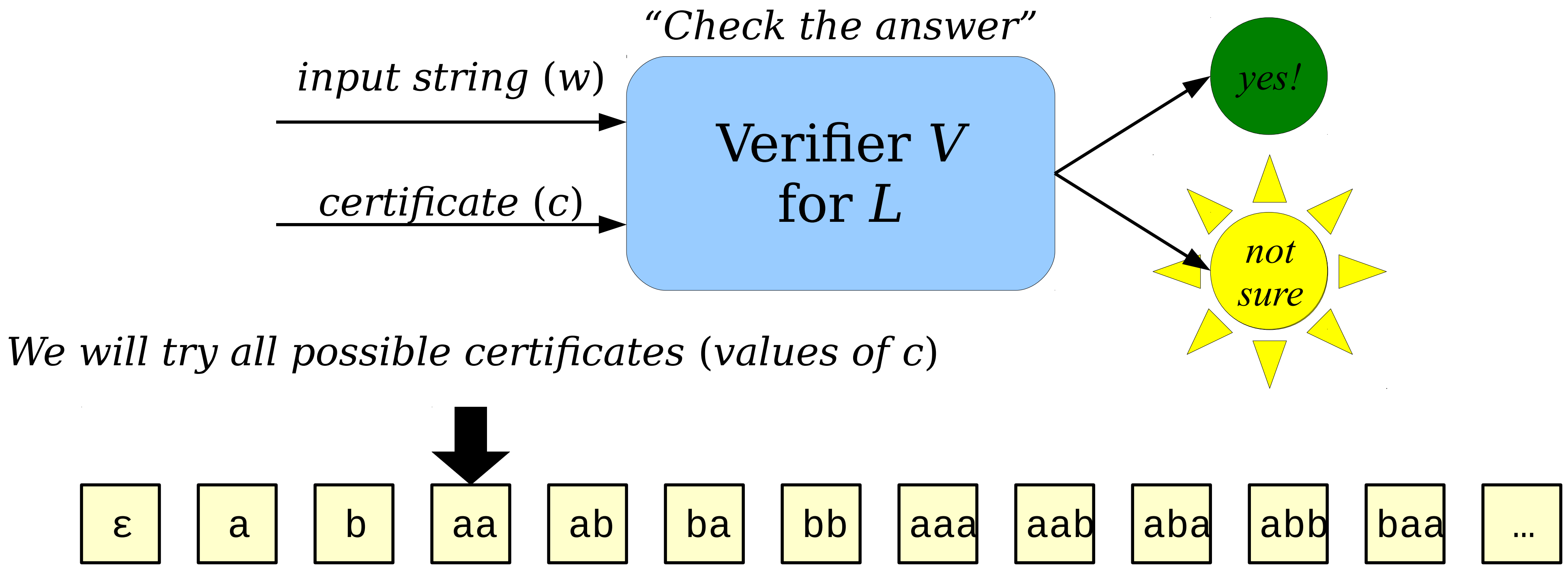

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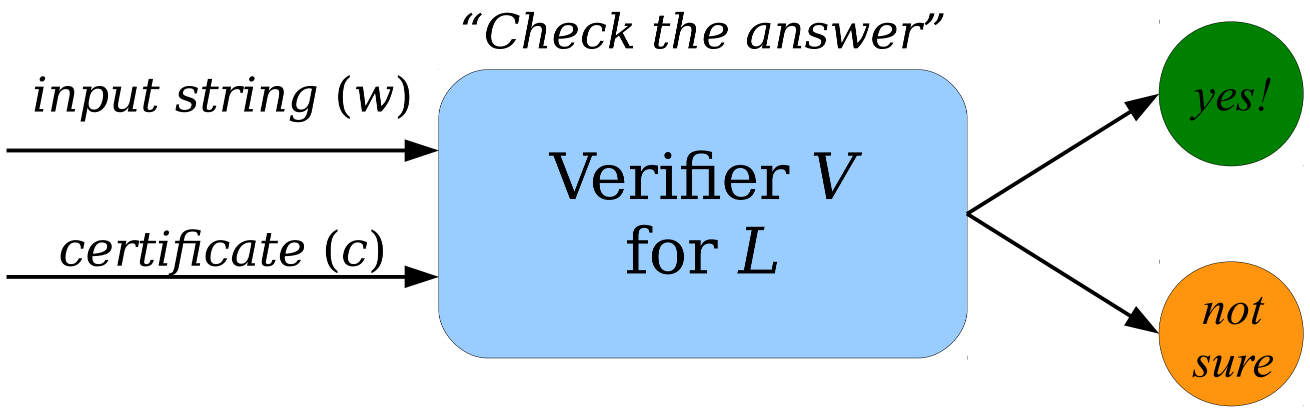

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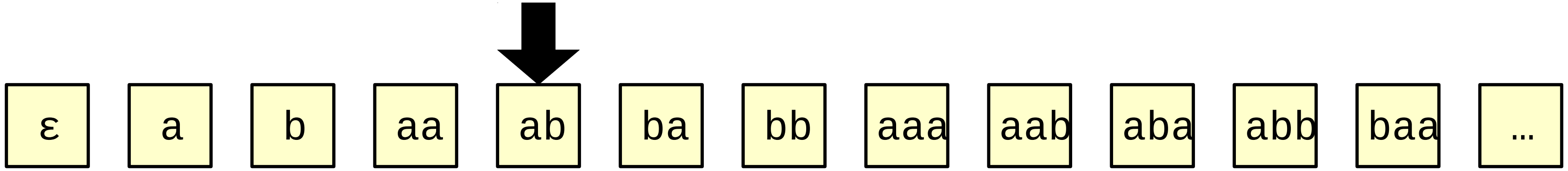

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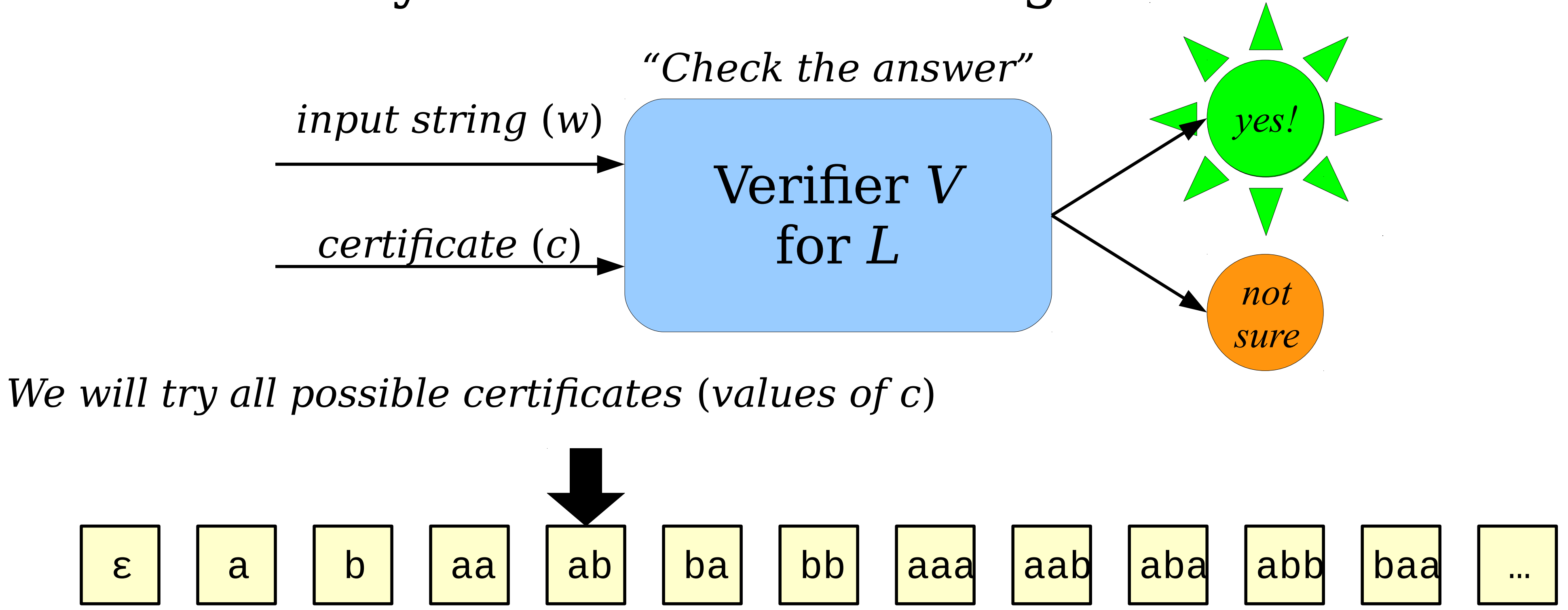

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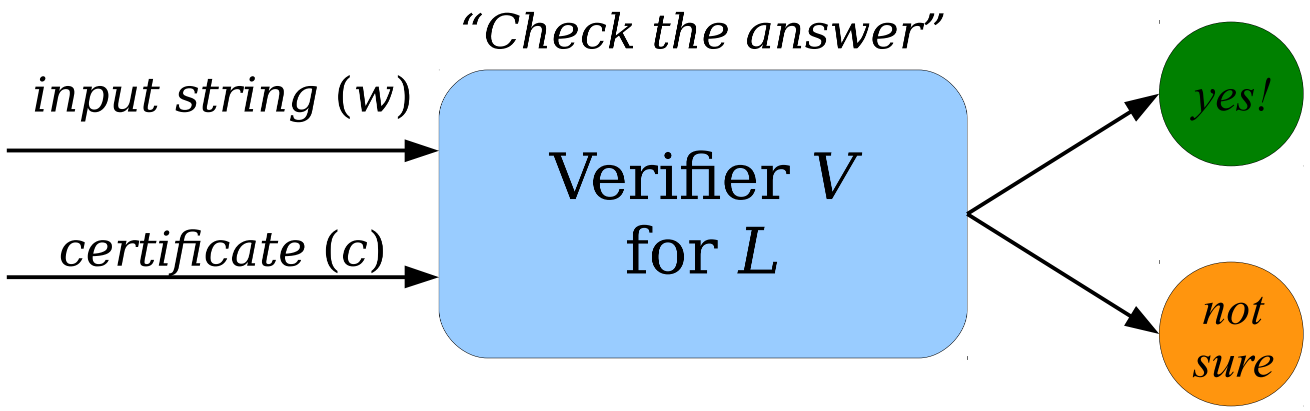

ε	a	b	aa	ab	ba	bb	aaa	aab	aba	abb	baa	…

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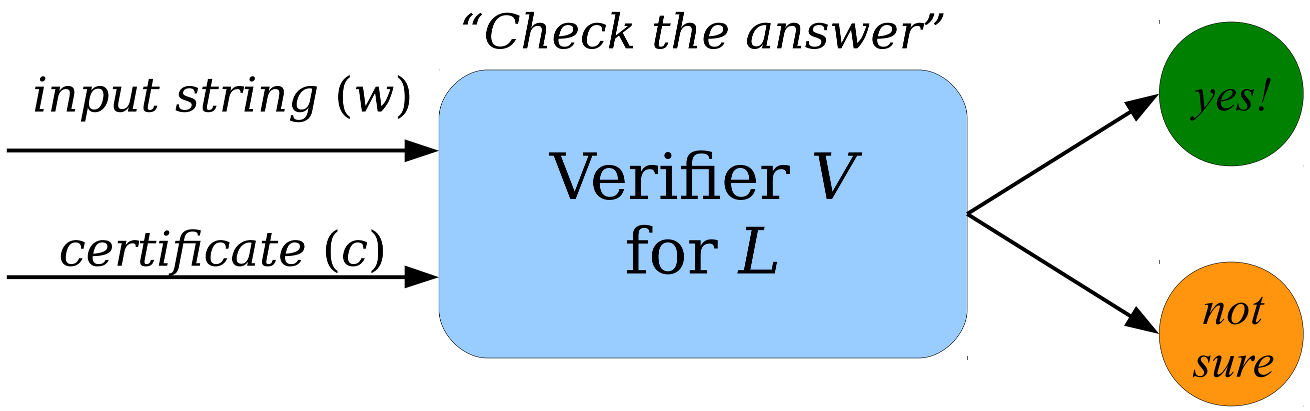

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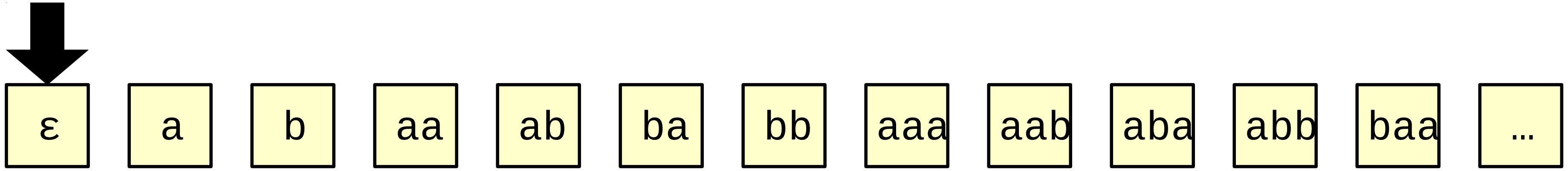

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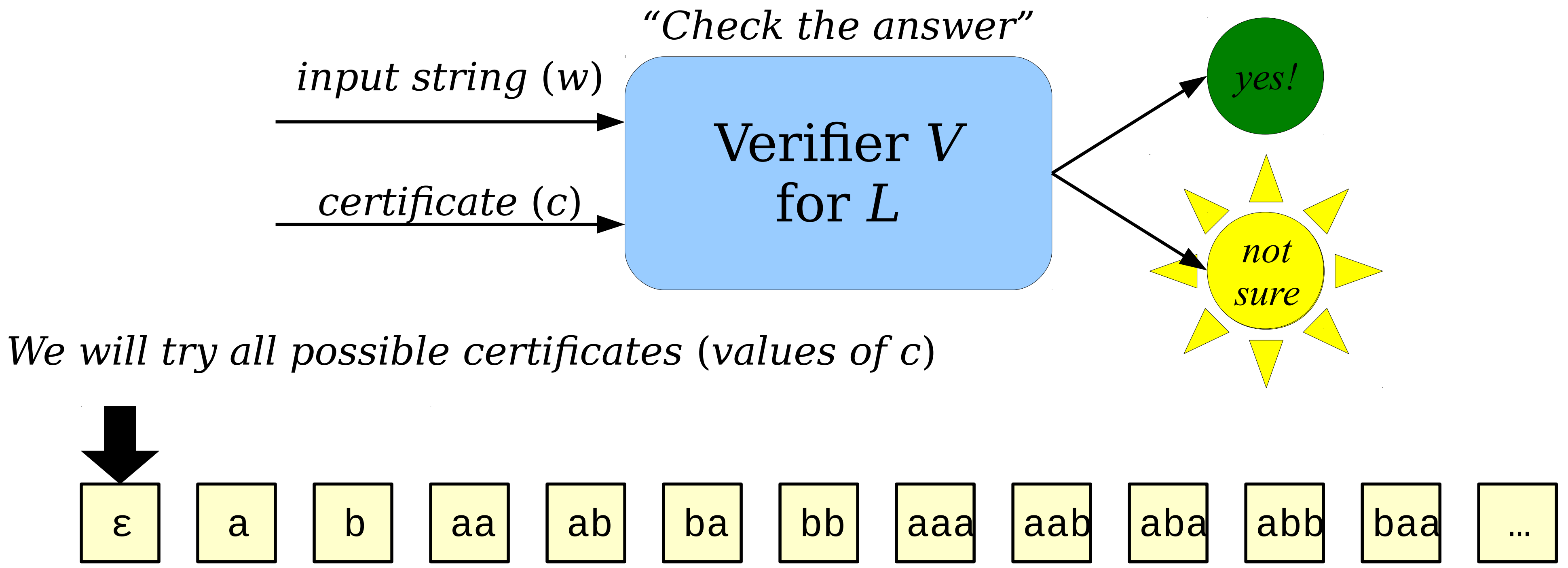

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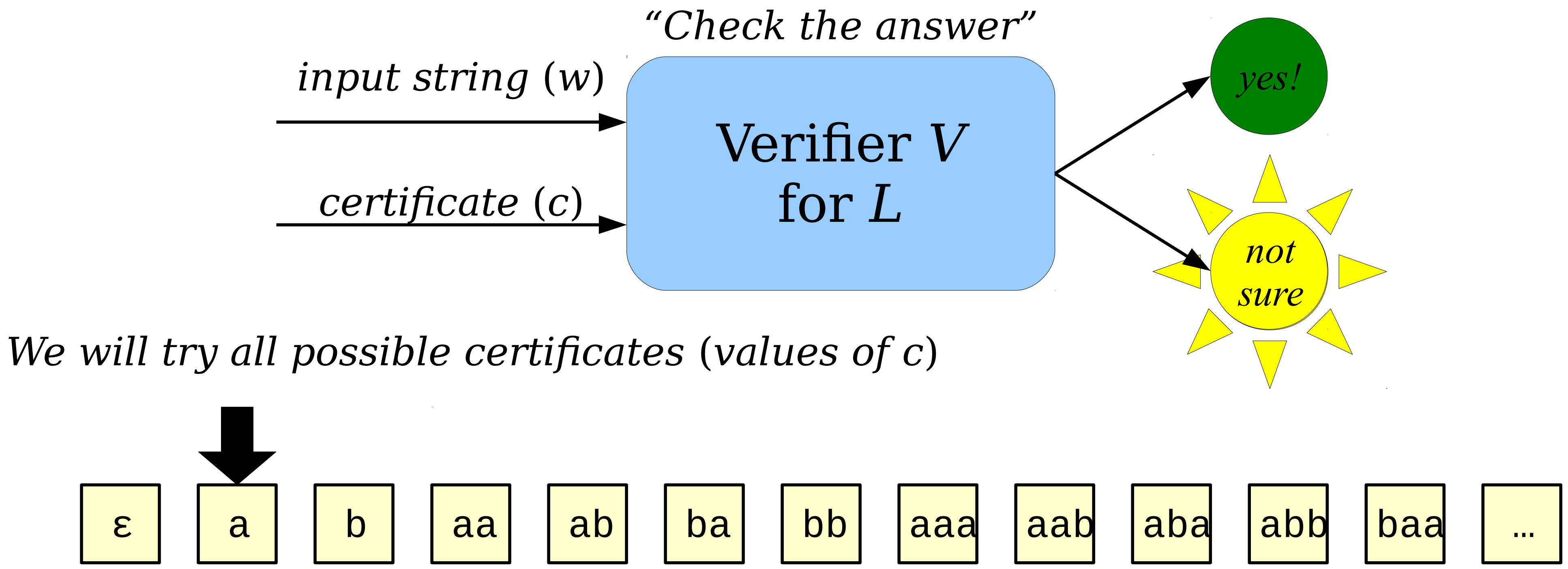

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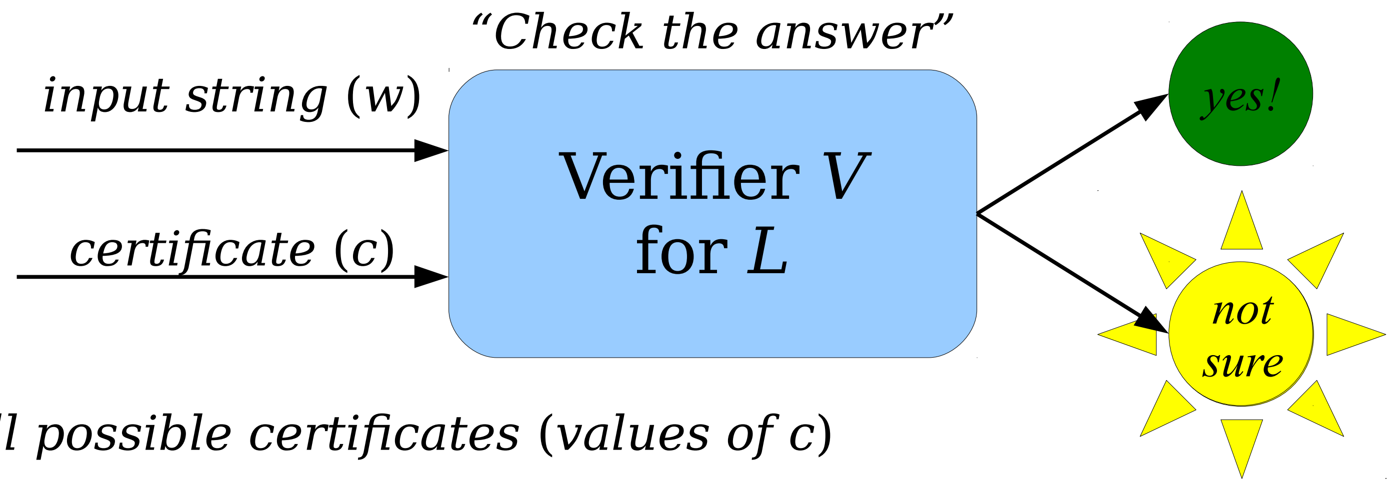

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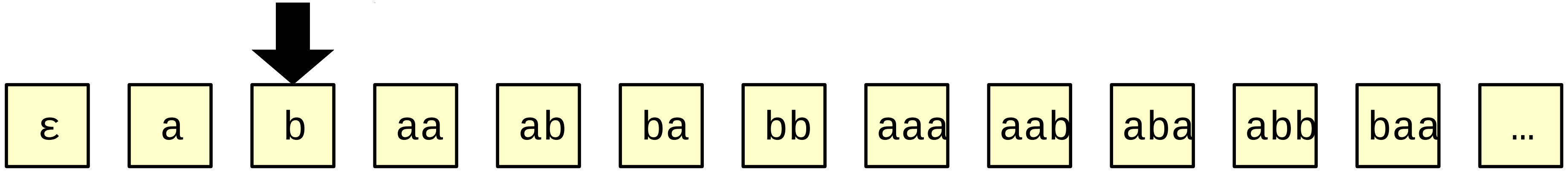

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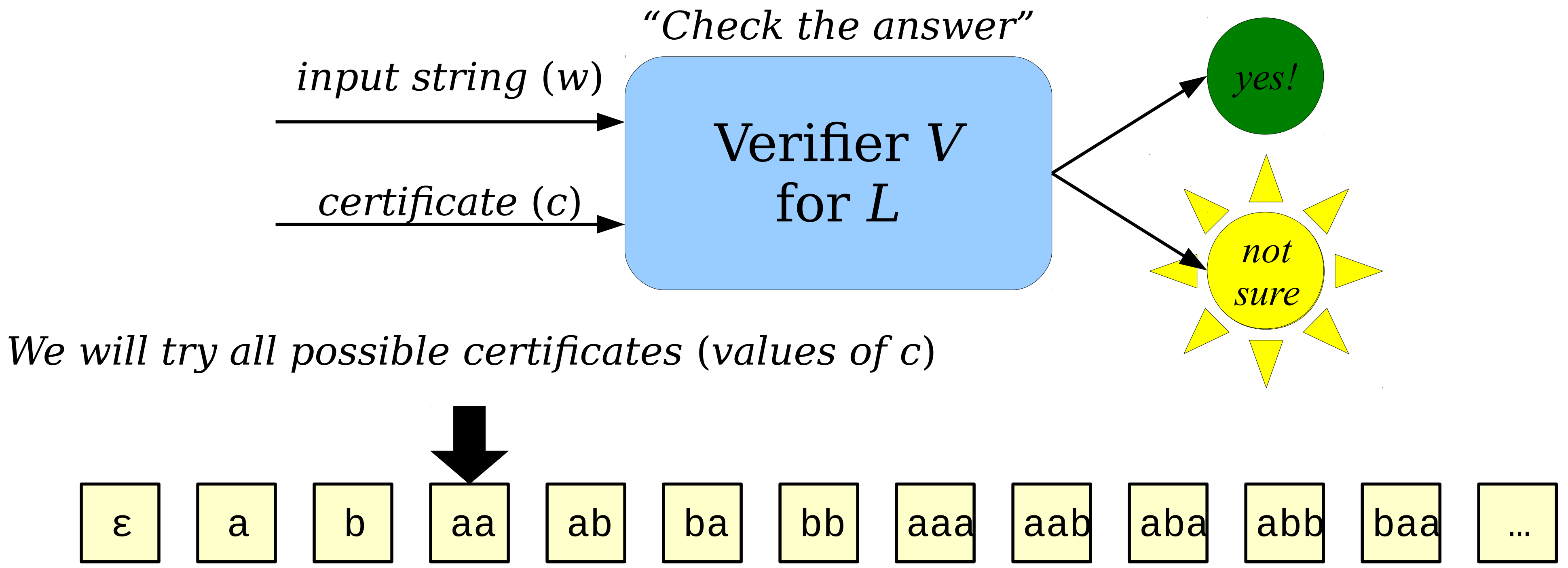

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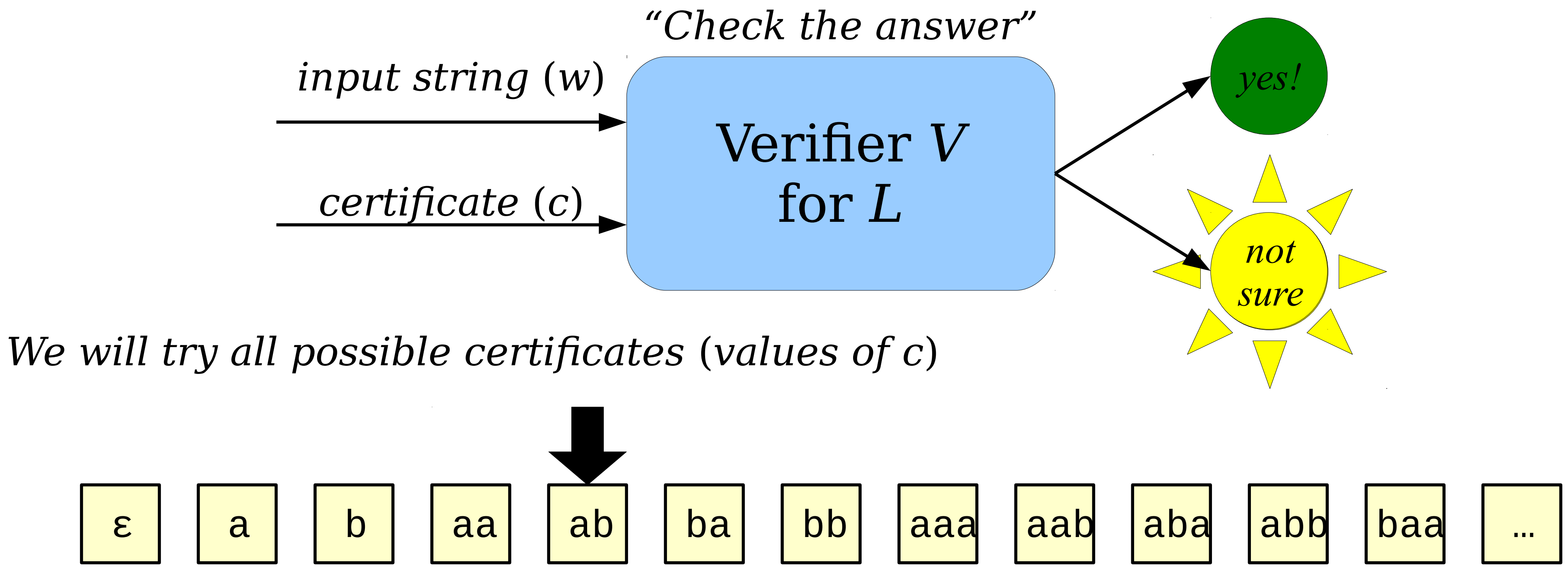

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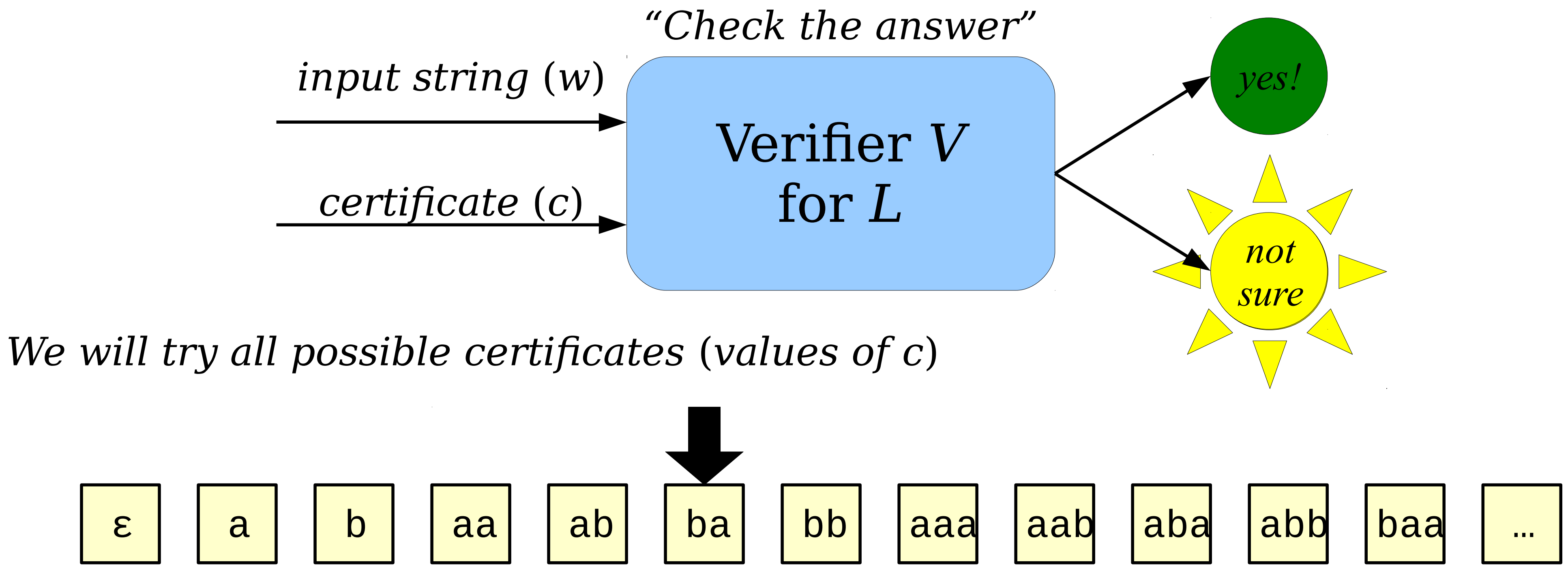

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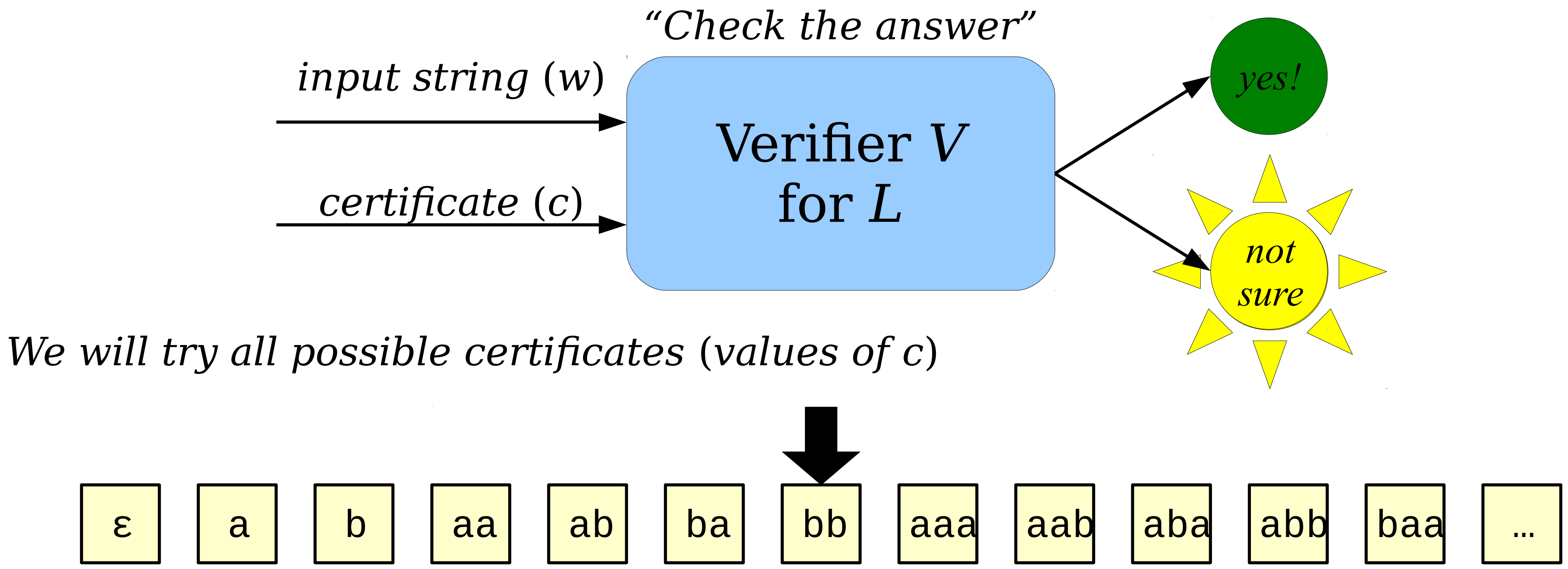

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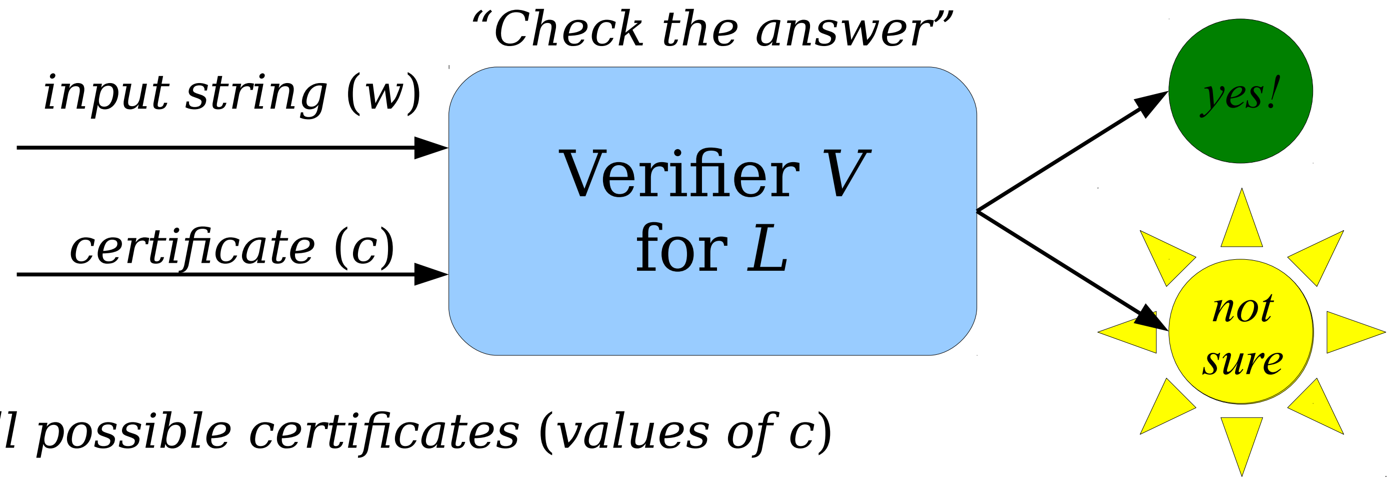

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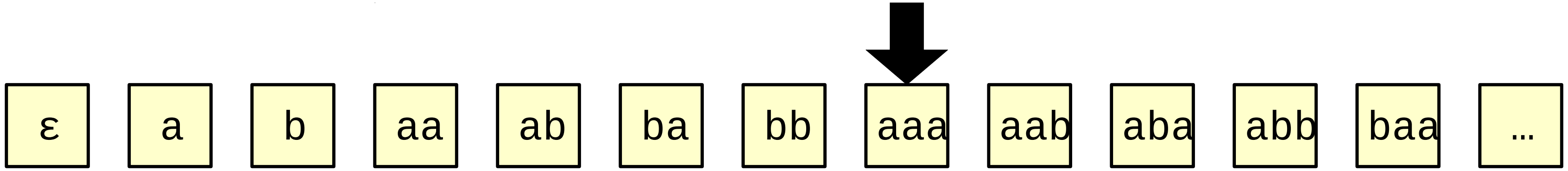

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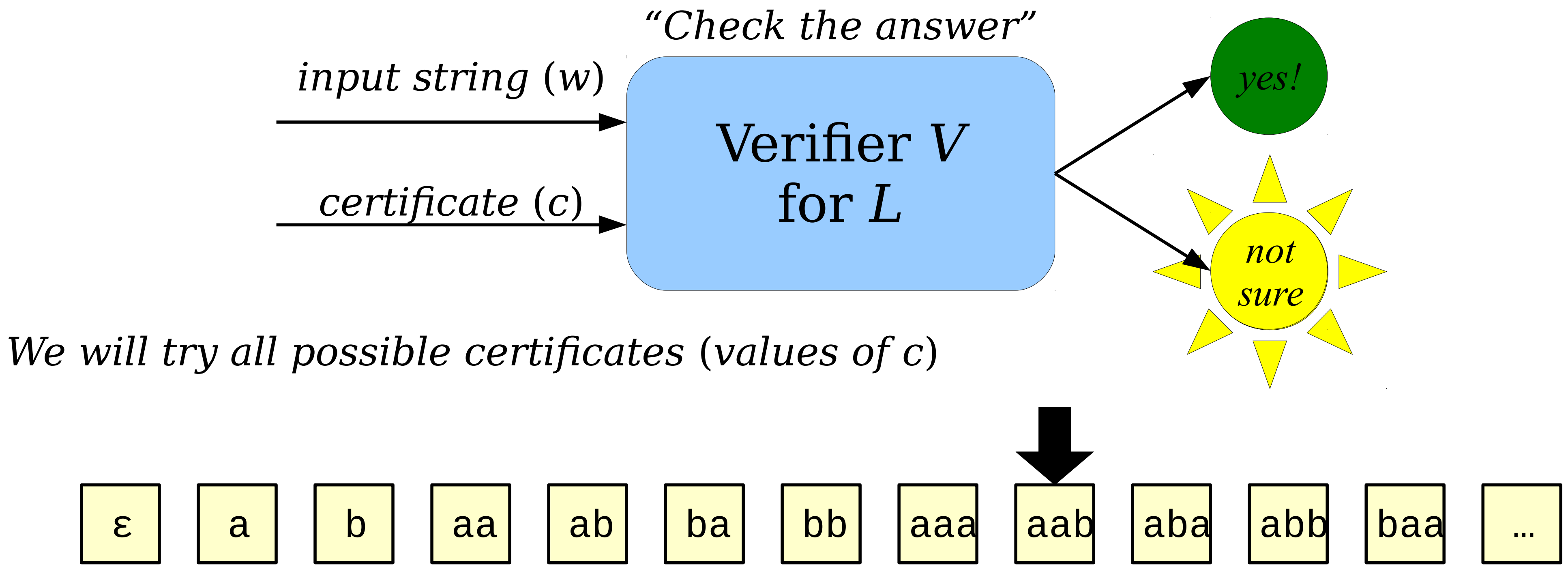

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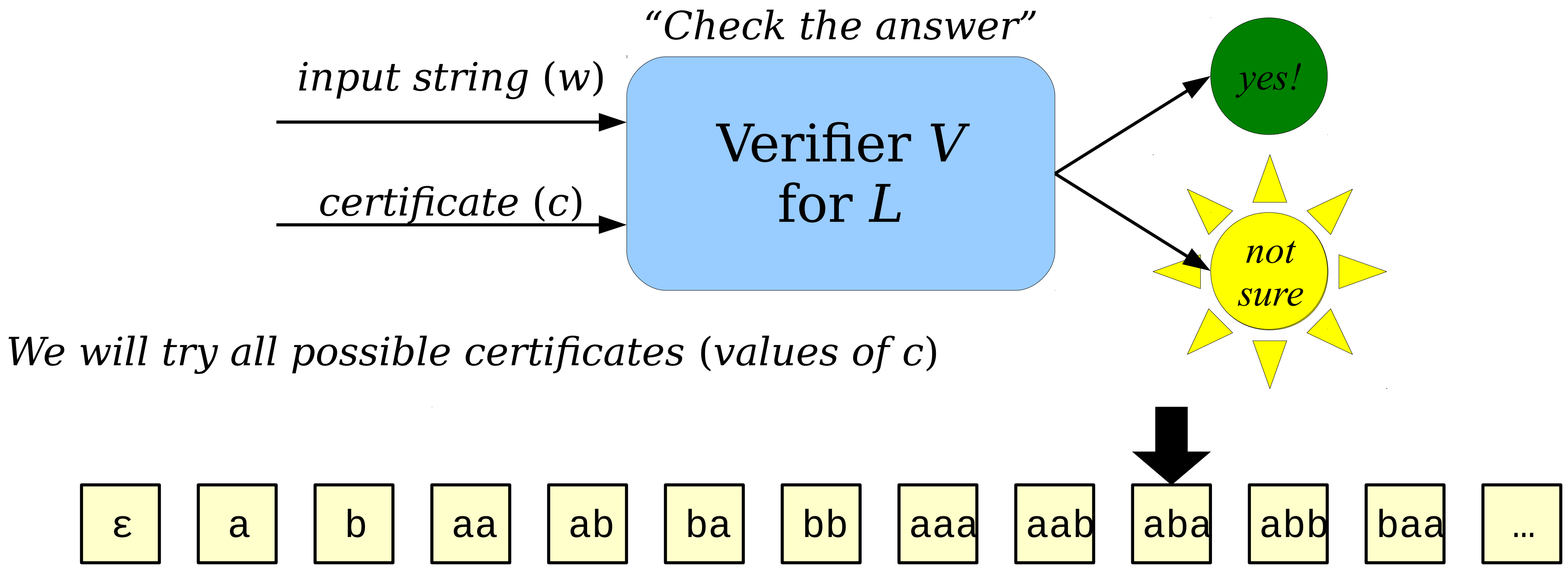

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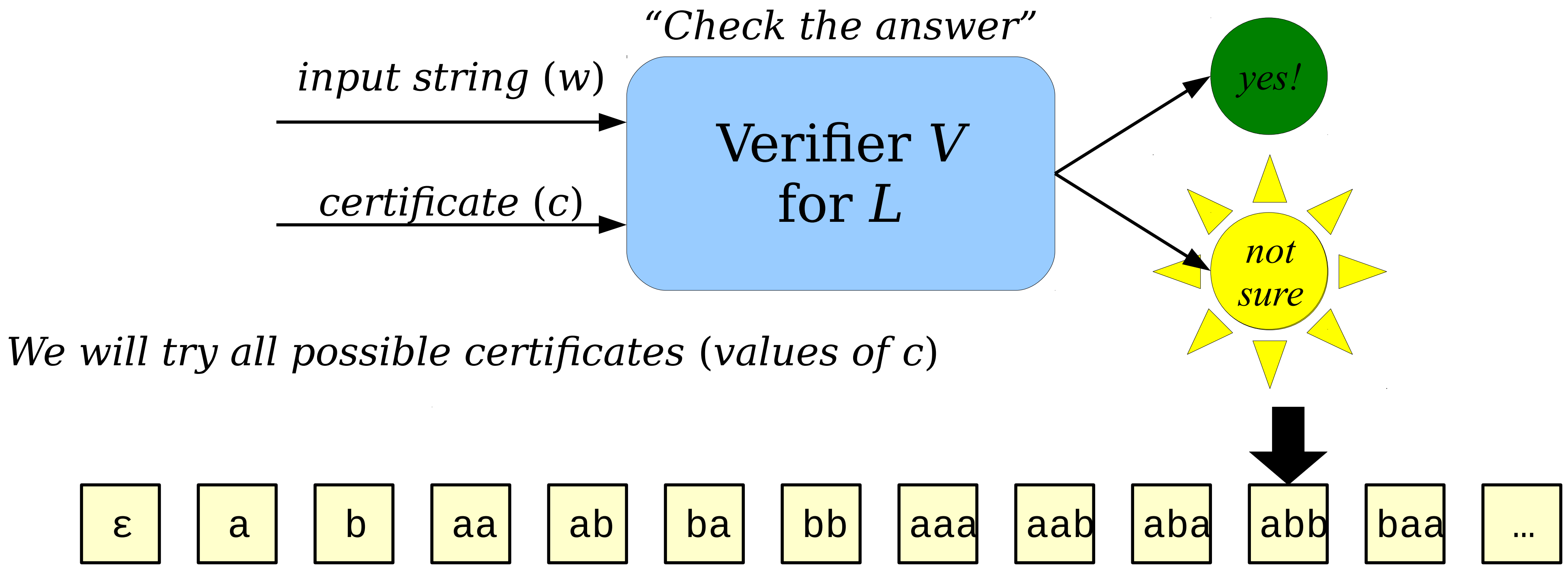

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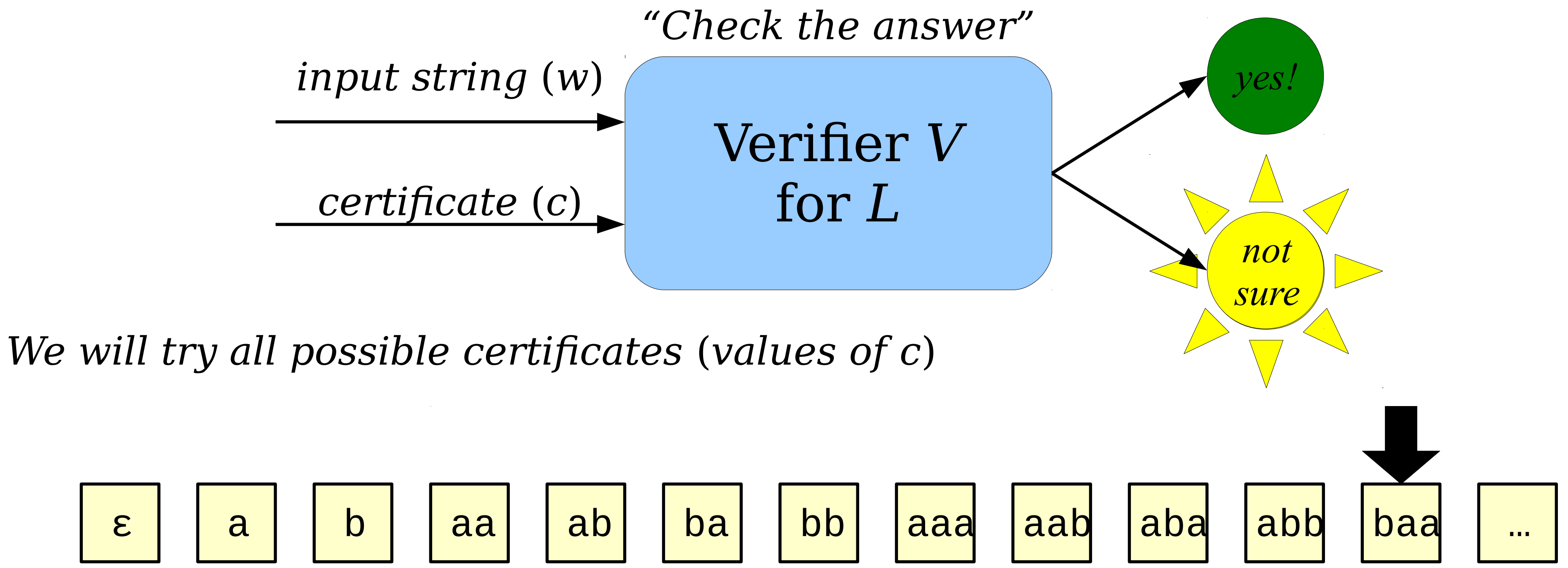

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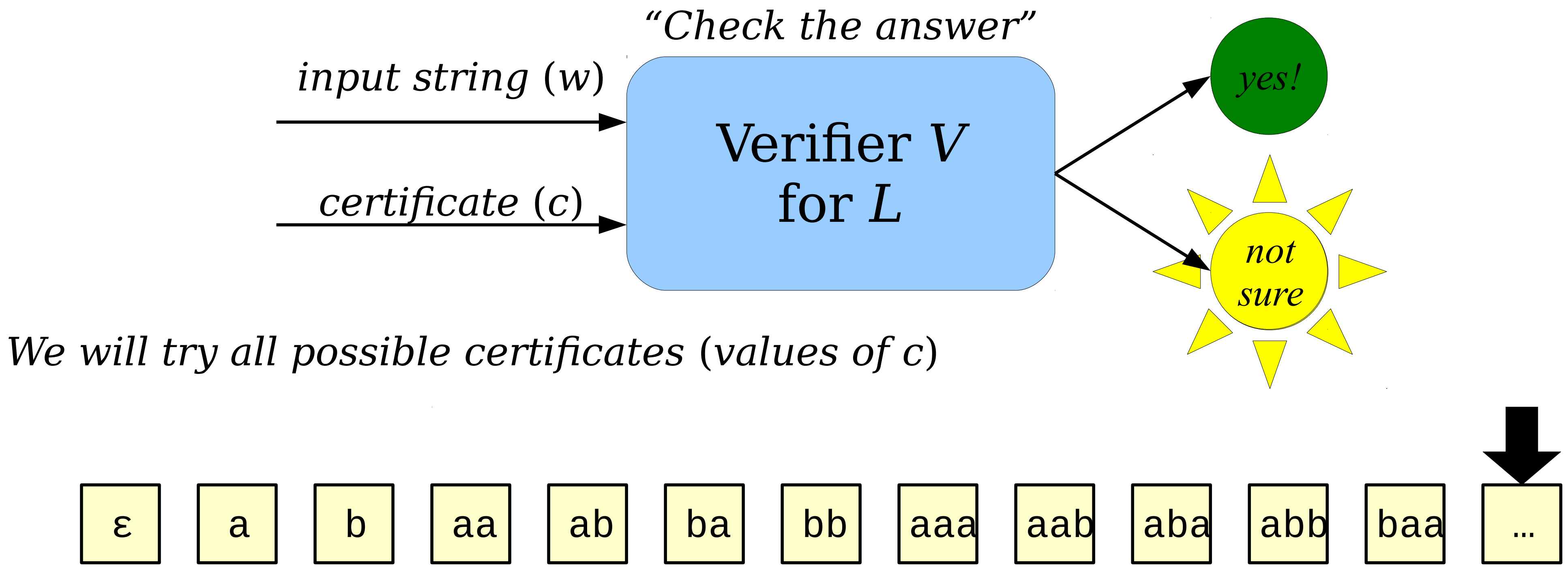


Verifiers and **RE**

- ***Theorem:*** If V is a verifier for L, then $L \in$ **RE**.
- ***Proof sketch:*** Consider the following program:

```
bool isInL(string w) {
   int i = 0;
   while (true) {
      for (each string c of length i) {
          if (V accepts ⟨w, c⟩) return true;
      }
      i++;
   }
}
```

If $w \in L$, there is some $c \in \Sigma^*$ where V accepts $\langle w, c\rangle$. The function `isInL` tries all possible strings as certificate, so it will eventually find c (or some other certificate), see V accept $\langle w, c\rangle$, then return true. Conversely, if `isInL(w)` returns true, then there was some string c such that V accepted $\langle w, c\rangle$, so $w \in L$. ■

Verifiers and **RE**

- ***Theorem:*** If $L \in \mathbf{RE}$, then there is a verifier for L.
- ***Proof goal:*** Beginning with a recognizer M for the language L, show how to construct a verifier V for L.
- The challenges:
  - A recognizer M is not required to halt on all inputs. A verifier V must always halt.
  - A recognizer M takes in one single input. A verifier V takes in two inputs.
- We'll need to find a way of reconciling these requirements.

Recall: If M is a recognizer for a language L, then M accepts w iff $w \in L$.

Key insight: If M accepts a string w, it always does so in a finite number of steps.

Idea: Adapt the verifier for A_{TM} into a more general construction that turns any recognizer into a verifier by running it for a fixed number of steps.

Verifiers and **RE**

- ***Theorem:*** If $L \in \mathbf{RE}$, then there is a verifier for L.
- ***Proof sketch:*** Consider the following program:

```
bool checkIsInL(string w, int c) {
   set up a simulation of M running on w;
   for (int i = 0; i < c; i++) {
       simulate the next step of M running on W;
   }
   return whether M is in an accepting state;
}
```

Notice that `checkIsInL` always halts, since each step takes only finite time to complete. Next, notice that if there is a c where `checkIsInL(w, c)` returns true, then M accepted w after running for c steps, so $w \in L$. Conversely, if $w \in L$, then M accepts w after some number of steps (call that number c). Then `checkIsInL(w, c)` will run M on w for c steps, watch M accept w, then return true. ■

RE and Proofs

- Verifiers and recognizers give two different perspectives on the "proof" intuition for **RE**.
- Verifiers are explicitly built to check proofs that strings are in the language.
  - If you know that some string w belongs to the language and you have the proof of it, you can convince someone else that $w \in L$.
- You can think of a recognizer as a device that "searches" for a proof that $w \in L$.
  - If it finds it, great!
  - If not, it might loop forever.

RE and Proofs

- If the **RE** languages represent languages where membership can be proven, what does a non-**RE** language look like?
- Intuitively, a language is *not* in **RE** if there is no general way to prove that a given string $w \in L$ actually belongs to L.
- In other words, even if you knew that a string was in the language, you may never be able to convince anyone of it!

Unsolvable Problems

Finding Non-**RE** Languages

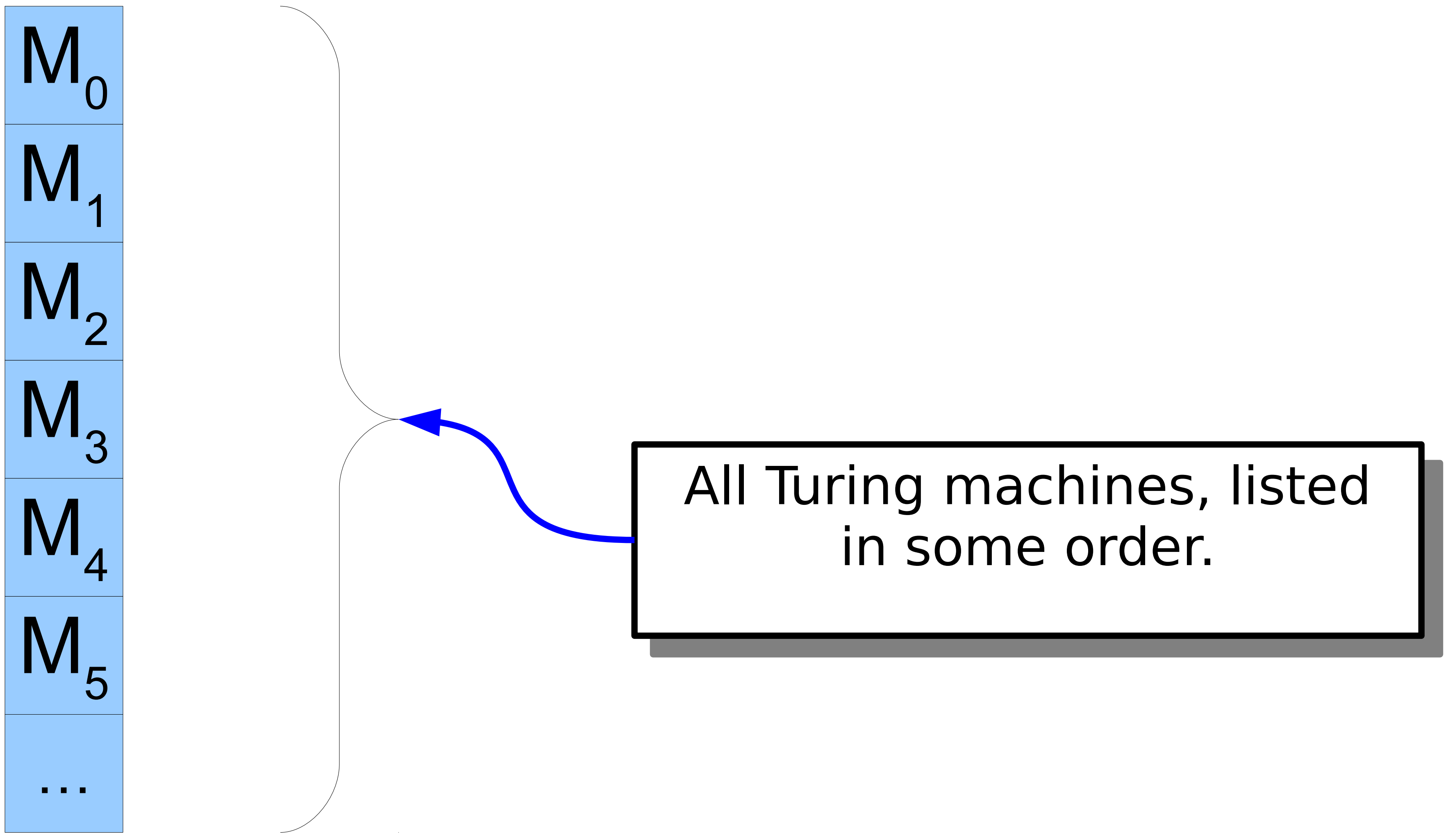
Finding Non-**RE** Languages

- Remember **RE** but non-**R** (undecidable) languages are those where we can reliably identify strings in the language, but cannot readily identify strings that are *not* in the language.
- Non-**RE** languages will be those where we cannot even readily identify strings that *are* in the language!
- How might we find an example of a non-**RE** language?

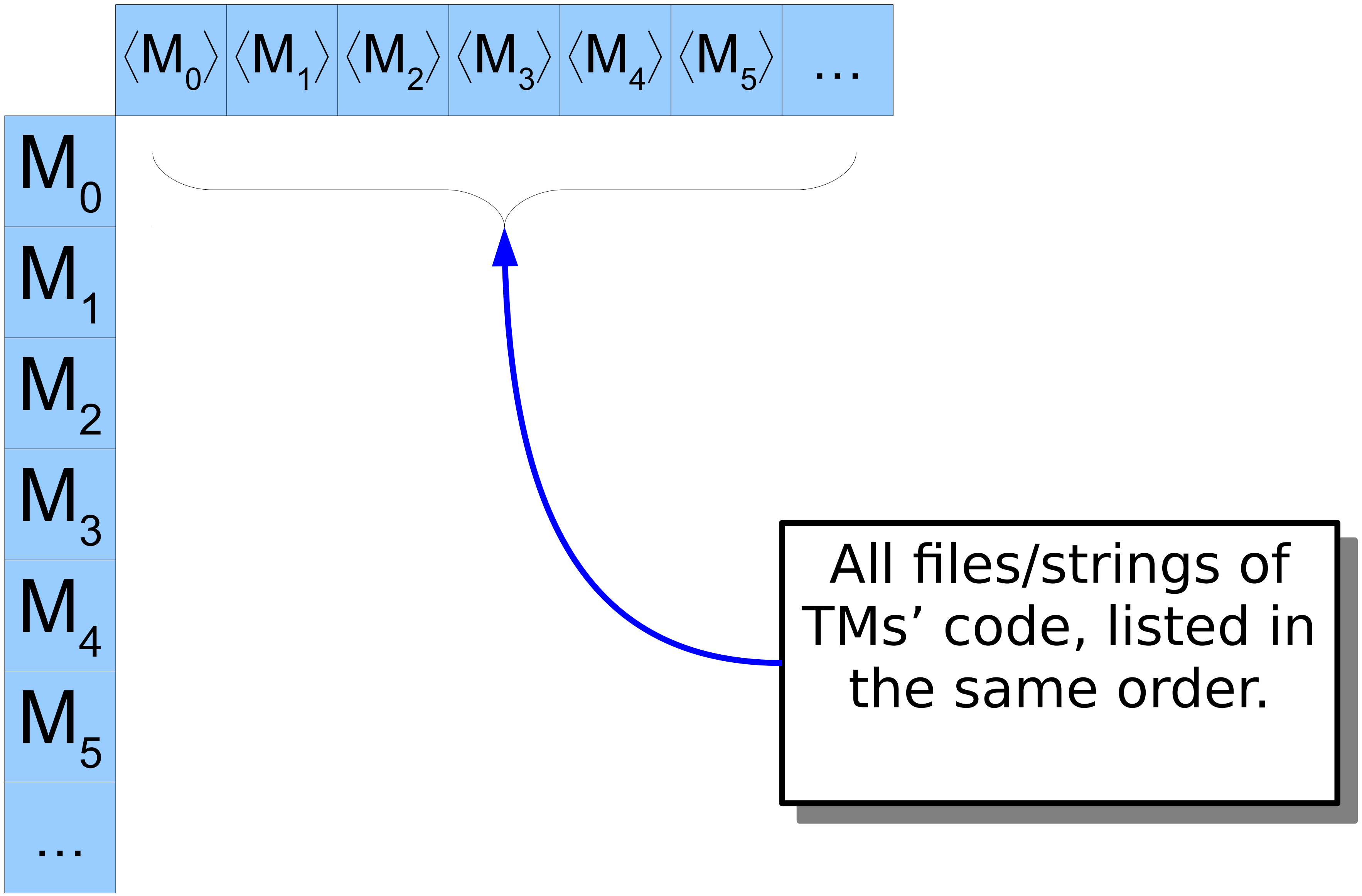
Languages, TMs, and TM Encodings

- What happens if we list off all Turing machines, looking at how those TMs behave when given other TM codes (as strings, so various $\langle M_x \rangle$ strings) as input?

M_0
M_1
M_2
M_3
M_4
M_5
…


M_0
M_1
M_2
M_3
M_4
M_5
...
All Turing machines, listed in some order.

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0							
M_1							
M_2							
M_3							
M_4							
M_5							
…							


⟨M0⟩ ⟨M1⟩ ⟨M2⟩ ⟨M3⟩ ⟨M4⟩ ⟨M5⟩ …
M0
M1
M2
M3
M4
M5
…
All files/strings of TMs’ code, listed in the same order.

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1							
M_2							
M_3							
M_4							
M_5							
…							

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2							
M_3							
M_4							
M_5							
…							

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3							
M_4							
M_5							
…							

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc		
M_3							
M_4							
M_5							
…							

What is $\mathscr{L}(M_0)$?

A. Σ^*

B. $\{\langle M_0 \rangle, \langle M_3 \rangle, \langle M_4 \rangle, \ldots\}$

C. $\{\langle M_0 \rangle\}$

D. $\{\langle M_0 \rangle, \langle M_1 \rangle, \langle M_2 \rangle, \langle M_3 \rangle, \ldots\}$

E. Something else.

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **A - E**.

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3							
M_4							
M_5							
…							

$\mathscr{L}(M_0) = \{\langle M_0 \rangle, \langle M_3 \rangle, \langle M_4 \rangle, \dots\}$. And we can't see the rest of the table for M_2, but it accepts everything so far, so it's at least possible that its language is $\mathscr{L}(M_2) = \Sigma^*$.

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc			
M_2	Acc	Acc	Acc	Acc			
M_3							
M_4							
M_5							
…							

Aside: we aren't really worrying about the existence of other strings that aren't TM code right now, but you could also think of it including those strings, so $\mathscr{L}(M_0) = \{\langle M_0 \rangle, \langle M_3 \rangle, \langle M_4 \rangle, \ldots\} \cup \{w \mid w \text{ is a string that isn't a TM's code, and } M_0 \text{ accepts } w\}$

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4							
M_5							
…							

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5							
…							

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…							

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…							

Quick check:
How many of the TMs on this chart so far do NOT accept their own code as a string?
(Enter a number.)

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **a number**.

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

What are we going to do next?

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **your answer**.

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

Acc	Acc	Acc	No	Acc	No	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

Flip all "accept" to "no" and vice-versa

No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

What TM has this behavior?

	⟨M_0⟩	⟨M_1⟩	⟨M_2⟩	⟨M_3⟩	⟨M_4⟩	⟨M_5⟩	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

	⟨M_0⟩	⟨M_1⟩	⟨M_2⟩	⟨M_3⟩	⟨M_4⟩	⟨M_5⟩	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…
	No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

No TM has this behavior!

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

“The language of all TMs that do not accept their own description.”

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

$\{ \langle M \rangle \mid M$ is a TM that does not accept $\langle M \rangle \}$

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

$$\{\ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathscr{L}(M)\ \}$$

Diagonalization Revisited

- The ***diagonalization language***, which we denote $\boldsymbol{L_{\mathrm{D}}}$, is defined as

$$\boldsymbol{L_{\mathrm{D}} = \{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathscr{L}(M) \}}$$

- That is, L_{D} is the set of descriptions of Turing machines that do not accept themselves.

$$L_D = \{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathscr{L}(M) \}$$

Theorem: $L_D \notin \mathbf{RE}$.

$L_D = \{ \langle M \rangle \mid M$ is a TM and $\langle M \rangle \notin \mathscr{L}(M) \}$

Theorem: $L_D \notin \mathbf{RE}$.

Proof: By contradiction; assume that $L_D \in \mathbf{RE}$.

$L_D = \{ \langle M \rangle \mid M$ is a TM and $\langle M \rangle \notin \mathscr{L}(M) \}$

Theorem: $L_D \notin \mathbf{RE}$.

Proof: By contradiction; assume that $L_D \in \mathbf{RE}$. Then there must be some recognizer R such that $\mathscr{L}(R) = L_D$.

$$L_D = \{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathscr{L}(M) \}$$

Theorem: $L_D \notin \mathbf{RE}$.

Proof: By contradiction; assume that $L_D \in \mathbf{RE}$. Then there must be some recognizer R such that $\mathscr{L}(R) = L_D$.

Let M be an arbitrary TM. Since $\mathscr{L}(R) = L_D$, we know that

$$\langle M \rangle \in L_D \text{ iff } \langle M \rangle \in \mathscr{L}(R). \qquad (1)$$

$$\boldsymbol{L_D = \{ \langle M \rangle \mid M \textbf{ is a TM and } \langle M \rangle \notin \mathscr{L}(M) \}}$$

Theorem: $L_D \notin$ **RE**.

Proof: By contradiction; assume that $L_D \in$ **RE**. Then there must be some recognizer R such that $\mathscr{L}(R) = L_D$.

Let M be an arbitrary TM. Since $\mathscr{L}(R) = L_D$, we know that

$$\langle M \rangle \in L_D \text{ iff } \langle M \rangle \in \mathscr{L}(R). \qquad (1)$$

Because $\mathcal{L}(R) = L_D$, we know that a string belongs to one set if and only if it belongs to the other.

$L_D = \{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathscr{L}(M) \}$

Theorem: $L_D \notin \mathbf{RE}$.

Proof: By contradiction; assume that $L_D \in \mathbf{RE}$. Then there must be some recognizer R such that $\mathscr{L}(R) = L_D$.

Let M be an arbitrary TM. Since $\mathscr{L}(R) = L_D$, we know that

$$\langle M \rangle \in L_D \quad \text{iff} \quad \langle M \rangle \in \mathscr{L}(R). \qquad (1)$$

From the definition of L_D, we see that $\langle M \rangle \in L_D$ iff $\langle M \rangle \notin \mathscr{L}(M)$.

$L_D = \{ \langle M \rangle \mid M$ is a TM and $\langle M \rangle \notin \mathscr{L}(M) \}$

Theorem: $L_D \notin \mathbf{RE}$.

Proof: By contradiction; assume that $L_D \in \mathbf{RE}$. Then there must be some recognizer R such that $\mathscr{L}(R) = L_D$.

Let M be an arbitrary TM. Since $\mathscr{L}(R) = L_D$, we know that

$$\langle M \rangle \in L_D \text{ iff } \langle M \rangle \in \mathscr{L}(R). \qquad (1)$$

From the definition of L_D, we see that $\langle M \rangle \in L_D$ iff $\langle M \rangle \notin \mathscr{L}(M)$. Combining this with statement (1) tells us that

$$\langle M \rangle \notin \mathscr{L}(M) \text{ iff } \langle M \rangle \in \mathscr{L}(R). \qquad (2)$$

$$L_D = \{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathscr{L}(M) \}$$

Theorem: $L_D \notin$ **RE**.

Proof: By contradiction; assume that $L_D \in$ **RE**. Then there must be some recognizer R such that $\mathscr{L}(R) = L_D$.

Let M be an arbitrary TM. Since $\mathscr{L}(R) = L_D$, we know that

$$\langle M \rangle \in L_D \text{ iff } \langle M \rangle \in \mathscr{L}(R). \qquad (1)$$

From the definition of L_D, we see that $\langle M \rangle \in L_D$ iff $\langle M \rangle \notin \mathscr{L}(M)$. Combining this with statement (1) tells us that

$$\langle M \rangle \notin \mathscr{L}(M) \text{ iff } \langle M \rangle \in \mathscr{L}(R). \qquad (2)$$

We've replaced the left-hand side of this biconditional with an equivalent statement.

$L_D = \{ \langle M \rangle \mid M$ is a TM and $\langle M \rangle \notin \mathscr{L}(M) \}$

Theorem: $L_D \notin \mathbf{RE}$.

Proof: By contradiction; assume that $L_D \in \mathbf{RE}$. Then there must be some recognizer R such that $\mathscr{L}(R) = L_D$.

Let M be an arbitrary TM. Since $\mathscr{L}(R) = L_D$, we know that

$$\langle M \rangle \in L_D \text{ iff } \langle M \rangle \in \mathscr{L}(R). \qquad (1)$$

From the definition of L_D, we see that $\langle M \rangle \in L_D$ iff $\langle M \rangle \notin \mathscr{L}(M)$. Combining this with statement (1) tells us that

$$\langle M \rangle \notin \mathscr{L}(M) \text{ iff } \langle M \rangle \in \mathscr{L}(R). \qquad (2)$$

Since our choice of M was arbitrary, we see that statement (2) holds for any TM M.

$L_D = \{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathscr{L}(M) \}$

Theorem: $L_D \notin \mathbf{RE}$.

Proof: By contradiction; assume that $L_D \in \mathbf{RE}$. Then there must be some recognizer R such that $\mathscr{L}(R) = L_D$.

Let M be an arbitrary TM. Since $\mathscr{L}(R) = L_D$, we know that

$$\langle M \rangle \in L_D \text{ iff } \langle M \rangle \in \mathscr{L}(R). \qquad (1)$$

From the definition of L_D, we see that $\langle M \rangle \in L_D$ iff $\langle M \rangle \notin \mathscr{L}(M)$. Combining this with statement (1) tells us that

$$\langle M \rangle \notin \mathscr{L}(M) \text{ iff } \langle M \rangle \in \mathscr{L}(R). \qquad (2)$$

Since our choice of M was arbitrary, we see that statement (2) holds for any TM M.

A nice consequence of a universally-quantified statement is that it should work in all cases.

$L_D = \{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathscr{L}(M) \}$

Theorem: $L_D \notin$ **RE**.

Proof: By contradiction; assume that $L_D \in$ **RE**. Then there must be some recognizer R such that $\mathscr{L}(R) = L_D$.

Let M be an arbitrary TM. Since $\mathscr{L}(R) = L_D$, we know that

$$\langle M \rangle \in L_D \quad \text{iff} \quad \langle M \rangle \in \mathscr{L}(R). \qquad (1)$$

From the definition of L_D, we see that $\langle M \rangle \in L_D$ iff $\langle M \rangle \notin \mathscr{L}(M)$. Combining this with statement (1) tells us that

$$\langle M \rangle \notin \mathscr{L}(M) \quad \text{iff} \quad \langle M \rangle \in \mathscr{L}(R). \qquad (2)$$

Since our choice of M was arbitrary, we see that statement (2) holds for any TM M. In particular, this means that statement (2) holds for the TM R, which tells us that

$$\langle R \rangle \notin \mathscr{L}(R) \quad \text{iff} \quad \langle R \rangle \in \mathscr{L}(R). \qquad (3)$$

$L_D = \{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathscr{L}(M) \}$

Theorem: $L_D \notin \mathbf{RE}$.

Proof: By contradiction; assume that $L_D \in \mathbf{RE}$. Then there must be some recognizer R such that $\mathscr{L}(R) = L_D$.

Let M be an arbitrary TM. Since $\mathscr{L}(R) = L_D$, we know that

$$\langle M \rangle \in L_D \quad \text{iff} \quad \langle M \rangle \in \mathscr{L}(R). \qquad (1)$$

From the definition of L_D, we see that $\langle M \rangle \in L_D$ iff $\langle M \rangle \notin \mathscr{L}(M)$. Combining this with statement (1) tells us that

$$\langle M \rangle \notin \mathscr{L}(M) \quad \text{iff} \quad \langle M \rangle \in \mathscr{L}(R). \qquad (2)$$

Since our choice of M was arbitrary, we see that statement (2) holds for any TM M. In particular, this means that statement (2) holds for the TM R, which tells us that

$$\langle R \rangle \notin \mathscr{L}(R) \quad \text{iff} \quad \langle R \rangle \in \mathscr{L}(R). \qquad (3)$$

This is clearly impossible.

$L_D = \{ \langle M \rangle \mid M$ is a TM and $\langle M \rangle \notin \mathscr{L}(M) \}$

Theorem: $L_D \notin \mathbf{RE}$.

Proof: By contradiction; assume that $L_D \in \mathbf{RE}$. Then there must be some recognizer R such that $\mathscr{L}(R) = L_D$.

Let M be an arbitrary TM. Since $\mathscr{L}(R) = L_D$, we know that

$$\langle M \rangle \in L_D \quad \text{iff} \quad \langle M \rangle \in \mathscr{L}(R). \qquad (1)$$

From the definition of L_D, we see that $\langle M \rangle \in L_D$ iff $\langle M \rangle \notin \mathscr{L}(M)$. Combining this with statement (1) tells us that

$$\langle M \rangle \notin \mathscr{L}(M) \quad \text{iff} \quad \langle M \rangle \in \mathscr{L}(R). \qquad (2)$$

Since our choice of M was arbitrary, we see that statement (2) holds for any TM M. In particular, this means that statement (2) holds for the TM R, which tells us that

$$\langle R \rangle \notin \mathscr{L}(R) \quad \text{iff} \quad \langle R \rangle \in \mathscr{L}(R). \qquad (3)$$

This is clearly impossible. We have reached a contradiction, so our assumption must have been wrong.

$L_D = \{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathscr{L}(M) \}$

Theorem: $L_D \notin \mathbf{RE}$.

Proof: By contradiction; assume that $L_D \in \mathbf{RE}$. Then there must be some recognizer R such that $\mathscr{L}(R) = L_D$.

Let M be an arbitrary TM. Since $\mathscr{L}(R) = L_D$, we know that

$$\langle M \rangle \in L_D \quad \text{iff} \quad \langle M \rangle \in \mathscr{L}(R). \qquad (1)$$

From the definition of L_D, we see that $\langle M \rangle \in L_D$ iff $\langle M \rangle \notin \mathscr{L}(M)$. Combining this with statement (1) tells us that

$$\langle M \rangle \notin \mathscr{L}(M) \quad \text{iff} \quad \langle M \rangle \in \mathscr{L}(R). \qquad (2)$$

Since our choice of M was arbitrary, we see that statement (2) holds for any TM M. In particular, this means that statement (2) holds for the TM R, which tells us that

$$\langle R \rangle \notin \mathscr{L}(R) \quad \text{iff} \quad \langle R \rangle \in \mathscr{L}(R). \qquad (3)$$

This is clearly impossible. We have reached a contradiction, so our assumption must have been wrong. Thus $L_D \notin \mathbf{RE}$.

$L_D = \{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathscr{L}(M) \}$

Theorem: $L_D \notin$ **RE**.

Proof: By contradiction; assume that $L_D \in$ **RE**. Then there must be some recognizer R such that $\mathscr{L}(R) = L_D$.

Let M be an arbitrary TM. Since $\mathscr{L}(R) = L_D$, we know that

$$\langle M \rangle \in L_D \quad \text{iff} \quad \langle M \rangle \in \mathscr{L}(R). \qquad (1)$$

From the definition of L_D, we see that $\langle M \rangle \in L_D$ iff $\langle M \rangle \notin \mathscr{L}(M)$. Combining this with statement (1) tells us that

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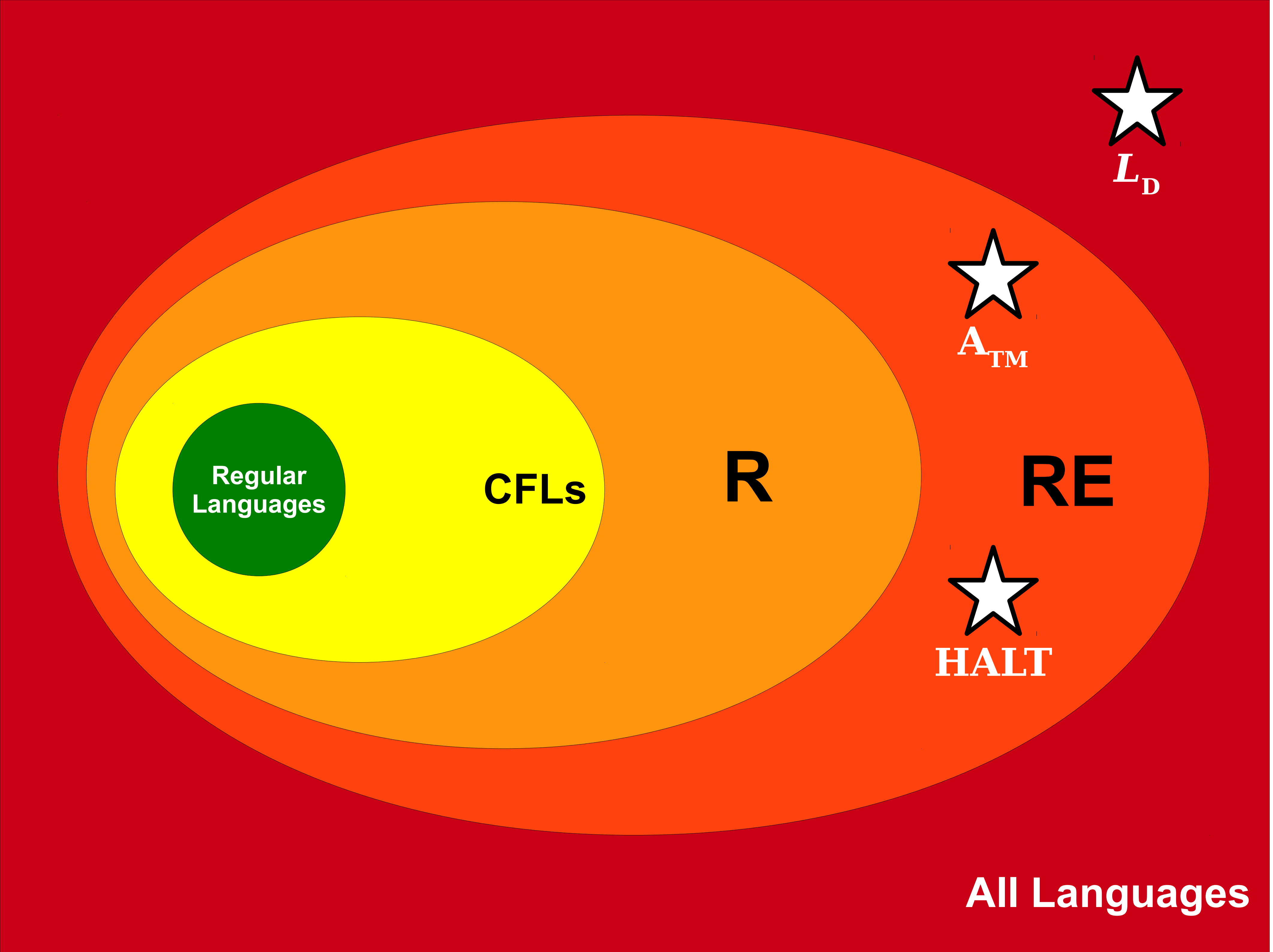

L_D
A_{TM}
Regular Languages
CFLs
R
RE
HALT
All Languages

What This Means

- On a deeper philosophical level, the fact that non-**RE** languages exist supports the following claim:

  There are statements that are true but not provable.

- Intuitively, given any non-**RE** language, there will be some string in the language that *cannot* be proven to be in the language.
- This result can be formalized as a result called ***Gödel's incompleteness theorem***, one of the most important mathematical results of all time.
- Want to learn more? Take Phil 152 or CS154!

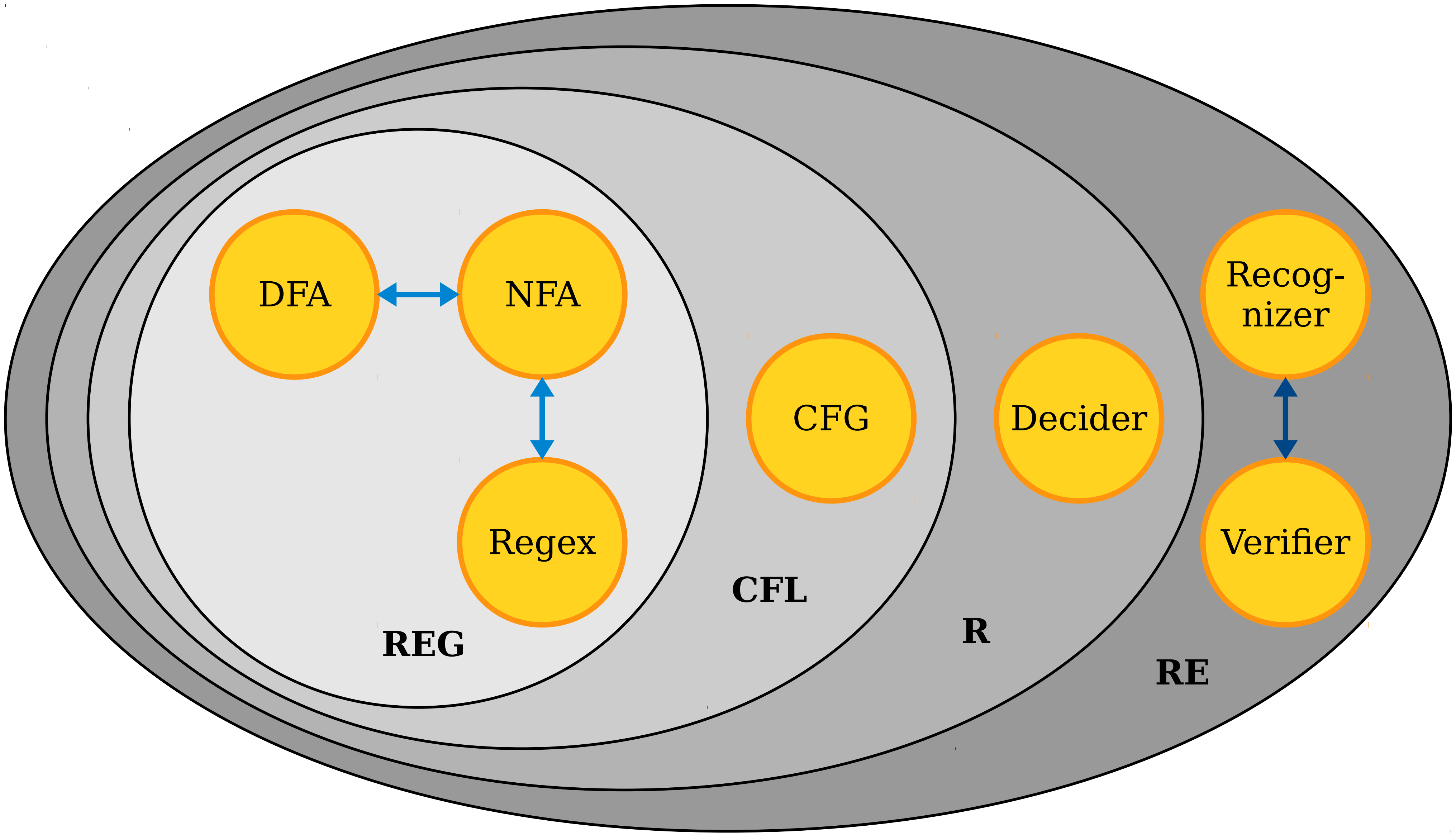
What This Means

- On a more philosophical note, you could interpret the previous result in the following way:

There are inherent limits about what mathematics can teach us.

- There's no automatic way to do math. There are true statements that we can't prove.
- That doesn't mean that mathematics is worthless. It just means that we need to temper our expectations about it.

The Big Picture



Up to this point:

"*Can* we solve this problem?"

(**Computability Theory**)

Up to this point:

“*Can* we solve this problem?”

(**Computability Theory**)

Up to this point:

"*Can* we solve this problem?"

(**Computability Theory**)

Starting today:

"Ok, even if we *can*, we need to consider whether the time/resources required actually make practical/feasible sense."

(**Complexity Theory**)

Where We've Been

- The class **R** represents problems that can be solved by a computer.
- The class **RE** represents problems where “yes” answers can be verified by a computer.

Where We're Going

- The class **P** represents problems that can be solved *efficiently* by a computer.
- The class **NP** represents problems where “yes” answers can be verified *efficiently* by a computer.